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ARRAY SIGNAL PROCESSING AND SPECTRUM ESTIMATION(US)  
POLYTECHNIC INST OF NEW YORK BROOKLYN DEPT OF  
ELECTRICAL ENGINEERING AND COMP UTER SCIENCE

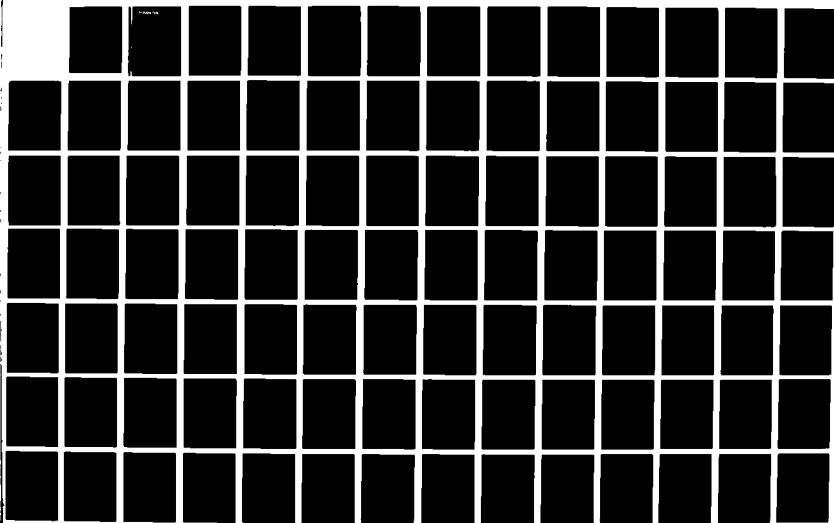
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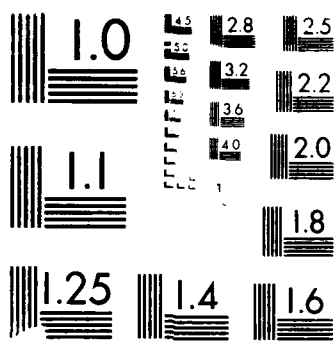
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## FINAL REPORT

TO  
OFFICE OF NAVAL RESEARCH

UNDER  
CONTRACT N00014-86-0321<sup>K</sup>

FOR RESEARCH ON  
ARRAY SIGNAL PROCESSING AND  
SPECTRUM ESTIMATION

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POLYTECHNIC UNIVERSITY  
DEPARTMENT OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE  
333 JAY STREET, BROOKLYN, NEW YORK 11201

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**SPECTRUM ESTIMATION**

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## 1. SUMMARY OF ACCOMPLISHMENTS

The major emphases of the research proposal submitted to the Office of Naval Research in 1986 were in the development of new techniques for coherent signal identification using multiple sensor elements and performance analysis of eigenstructure-based high resolution methods for direction finding.

This research report concerns primarily with new developments (since the last report date) in the detection of signals from measurements made at the outputs of a set of spatially deployed sensor elements. Three key results that have been obtained during this investigation is reported in section 2 with all details. To be specific, they are:

In the context of coherent signal classification, a spatial smoothing scheme first suggested by Evans *et al.*, and subsequently studied by Shan *et al.*, is further investigated. It is proved in section 2.1 that by making use of a set of forward and complex conjugated backward subarrays simultaneously, it is always possible to estimate any  $K$  directions of arrival using at most  $3K/2$  sensor elements. This is achieved by creating a smoothed array output covariance matrix that is structurally identical to a covariance matrix in some noncoherent situation. By incorporating the eigenstructure-based techniques on this smoothed covariance matrix, it then becomes possible to correctly identify all directions of arrivals irrespective of their correlation.

Section 2.2 presents a detailed asymptotic analysis of a class of high resolution estimators for resolving correlated and coherent plane waves in noise. These estimators are in turn constructed from certain eigenvectors associated with spatially smoothed (or unsmoothed) covariance matrices generated from a uniform array. The analysis is first carried out for the smoothed case, and from this the conventional MUSIC (unsmoothed) scheme follows as a special case. Independent of the total number of sources present in the scene, the variance of the conventional MUSIC esti-

nator along the true arrival angles is shown to be zero within a first-order approximation. Further, the bias expressions in the smoothed case are used to obtain a resolution threshold for two coherent, equipowered plane wave sources in white noise, and the result is compared to that obtained for two uncorrelated, equipowered plane wave sources.

To the best of our knowledge the analysis presented in section 2.2 is new and should turn out to be extremely useful for the performance evaluation of almost all eigenstructure based techniques.

Finally, a technique for estimating the directions of arrival of multiple signals utilizing the generalized eigenvalues associated with certain matrices generated from the signal subspace eigenvectors is proposed in section 2.3. This is carried out by observing a well known property of the signal subspace; i.e., in presence of uncorrelated and identical sensor noise, the subspace spanned by the true direction vectors coincides with the one spanned by the eigenvectors corresponding to all, except the smallest set of repeating eigenvalues of the array output covariance matrix. Further, utilizing the results in section 2.2, a first-order perturbation analysis is carried out to evaluate the performance of this scheme, when the array output cross-covariances are estimated from the data. In its least favorable configuration, unlike the MUSIC scheme, the bias associated with the directions-of-arrival estimator for this scheme in a two-source scene is shown to be zero and the variance to be nonzero within a first-order approximation. In its most favorable configuration, both bias and variance are shown to be nonzero for the same source scene. Using these variance expressions, resolution thresholds are obtained for two closely spaced sources in an equipowered scene. The superior performance of this algorithm in its most favorable configuration is also shown to be in agreement with actual simulation results.

A list of publications originated under this contract is outlined in the next section.

## 1.1 List of Publications

### Papers submitted to Journals

- [1] David Pearson, S. Unnikrishna Pillai, and Youngjik Lee, "An algorithm for near-optimal placement of sensor elements," submitted to *IEEE Trans. Inform. Theory*.
- [2] Byung Ho Kwon, S. Unnikrishna Pillai and Rabindra N. Madan, "GEESE (GEneralized Eigenvalues utilizing signal Subspace Eigenvectors) - new performance results on direction finding," submitted to *IEEE Trans. Acoustics, Speech, Signal Processing*.

### Journal Papers published

- [3] S. Unnikrishna Pillai and Fred Haber, "Statistical analysis of a high resolution spatial spectrum estimator utilizing an augmented covariance matrix," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 1517-1523, Nov. 1987.
- [4] S. Unnikrishna Pillai and Byung Ho Kwon, "Forward/backward spatial smoothing techniques for coherent signal identification," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-37, no. 1, pp. 8-15, Jan. 1989.
- [5] S. Unnikrishna Pillai and Youngjik Lee, "Coherent signal classification using symmetry considerations," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-37, no. 1, pp. 135-138, Jan. 1989.
- [6] Byung Ho Kwon and S. Unnikrishna Pillai, "A self inversive array processing scheme for angle-of-arrival estimation," to appear in *European Signal Processing Journal*, July 1989.
- [7] S. Unnikrishna Pillai and Byung Ho Kwon, "Performance analysis of MUSIC-type high resolution estimators for direction finding in correlated and coherent scenes," to appear in *IEEE Trans. Acoust., Speech, Signal Processing*, Aug. 1989.

### Conference Publications

- [8] S. Unnikrishna Pillai and Fred Haber, "A new test in factor analysis based on higher powers of the sample eigenvalues," presented at the 1986 IEEE International Symposium on Information Theory, University of Michigan, Ann Arbor, October 5-9, 1986.
- [9] S. Unnikrishna Pillai and Fred Haber, "Statistical analysis of a high resolution spatial spectrum estimator utilizing an augmented covariance matrix," Proceedings of the third ASSP Workshop on Spectrum Estimation and Modeling, Northeastern University, Boston, November 17-18, 1986.



- [10] S. Unnikrishna Pillai and Fred Haber, "Can extra unknown parameters hurt?" presented at the fifth workshop on Multidimensional Signal Processing (IEEE ASSP & EURASIP), The Delft, Netherlands, September 14-16, 1987.
- [11] S. Unnikrishna Pillai, "Coherent signal classification by symmetry considerations," presented at Conference on Information Systems and Sciences, Johns Hopkins University, Baltimore, March 25-27, 1987.
- [12] Youngjik Lee and S. Unnikrishna Pillai, "Spatial spectrum estimation: a nonparametric approach," presented at the 30th Midwest Symposium on Circuits and Systems, Syracuse University, Syracuse, August 15-17, 1987.
- [13] Byung Ho Kwon and S. Unnikrishna Pillai, "A self inversive array processing scheme for improved performance," presented at the 22nd Annual Conference on Information Science and Systems, Princeton, NJ, March 16-18, 1988.
- [14] Youngjik Lee and S. Unnikrishna Pillai, "An algorithm for optimal placement of sensor elements," presented at ICASSP-88, New York, NY, April 11-14, 1988.
- [15] Byung Ho Kwon and S. Unnikrishna Pillai, "Performance analysis of eigenvector-based high resolution estimators for direction finding in correlated and coherent scenes," Fourth ASSP Workshop on Spectrum Estimation and Modeling, University of Minnesota, Aug. 3-5, 1988.
- [16] S. Unnikrishna Pillai, Youngjik Lee and D. C. Youla, "Angle of arrival estimation based on first order statistics," presented at the 1988 Digital Signal Processing Workshop, Lake Tahoe, CA, September 14-17, 1988.
- [17] S. Unnikrishna Pillai and B. H. Kwon, "GEESE (GEneralized Eigenvalues utilizing Signal subspace Eigenvectors) - A new technique for direction finding," presented at the Twenty second Annual Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA, Oct. 31 - Nov. 2, 1988.
- [18] Rabindra N. Madan and S. Unnikrishna Pillai, "Statistics of generalized eigenvalues for signal parameter estimation," presented at the second IMA conference on Mathematics in Signal Processing, University of Warwick, England, Dec. 13-15, 1988.

#### **Graduate Student(s) Supported**

##### **1. Youngjik Lee**

Full time Ph. D. student and research fellow under this contract from June 1986 to December 1988. Completed his Ph.D. degree in December, 1988.

Ph. D. dissertation title: "Direction Finding from First Order Statistics and Spatial Spectrum Estimation".

##### **2. Byung Ho Kwon**

Full time Ph.D. Student and research fellow under this contract from January 1988 to December 1988.

Ph. D. dissertation title: "New High Resolution Techniques and Their Performance Analysis for Angles-of-Arrival Estimation".

**Electrical Engineering Seminars presented by the P 1**

1. Array Signal Processing, Stanford University, CA, October 2<sup>nd</sup> 1988.
2. 'Image Reconstruction from Signal Magnitude: A Uniqueness Characterization for Functions in the Polychain Algebra,' Stanford University, CA, October 28, 1988.
3. 'Recent Advances in High Resolution Direction Finding and Their Performance Analysis,' Imperial College, London, December 5, 1988.

## TECHNICAL REPORT

### 2.1 Forward/Backward Spatial Smoothing Techniques for Coherent Signal Identification

#### 2.1.1 Introduction

In recent years, considerable effort has been spent in developing high resolution techniques for estimating the directions of arrival of multiple signals using multiple sensors. These methods [1]–[4], in general, exploit specific eigenstructure properties of the sensor array output covariance matrix and are known to yield high resolution even when the signal sources are partially correlated. However, when some of the signals are perfectly correlated (coherent), as happens, for example, in multipath propagation, these techniques encounter serious difficulties. Several alternatives have been proposed [5]–[11] to take care of this situation, of which the spatial smoothing scheme first suggested by Evans *et al.* [9], [10] and extensively studied by Shan *et al.* [11], [12] is specially noteworthy. Their solution is based on a preprocessing scheme that partitions the total array of sensors into subarrays and then generates the average of the subarray output covariance matrices. Shan *et al.* have shown that when this average of subarray covariance matrices is used in conjunction with the eigenstructure-based multiple signal classification technique developed by Schmidt [3], in the case of independent and identical sensor noise, it is possible to estimate all directions of arrival irrespective of their degree of correlation. However, this forward-only smoothing scheme makes use of a larger number of sensor elements than the conventional ones, and in particular requires  $2K$  sensor elements to estimate any  $K$  directions of arrival.

In this report, we analyze an improved spatial smoothing scheme –called the forward/backward smoothing scheme– and prove that at most  $[3K/2]^{(1)}$  elements are

enough to estimate any  $K$  directions of arrival. In addition to the forward subarrays, this scheme makes use of complex conjugated backward subarrays of the original array to achieve superior performance. In this context, it is instructive to note the observations of Evans *et al.* [10], "The combined effect of spatial smoothing and forward/backward averaging cannot increase an array's direction finding capability beyond  $[2M/3]$  coherent signals (with  $M$  representing the number of sensor elements)." While this statement is correct and coincides with the bounds in [13], Evans *et al.* do not provide a proof for it. A special case of the general situation, where the multipath coefficients are treated to be real, is proved in [13]. However, this is an unrealistic assumption, as in practice all multipath coefficients will be invariably complex numbers and in that case it is necessary to reason differently.

For clarity of presentation, section 2.1.2 deals with a completely coherent situation and proves that to estimate any  $K$  coherent directions of arrival it is sufficient to have an array of  $[3K/2]$  sensors. The proof for the general non-coherent case is sketched in the Appendix.

### 2.1.2 Direction Finding in a Coherent Environment

Consider a uniform linear array consisting of  $M$  identical sensors and receiving signals from  $K$  narrowband coherent signals that arrive at the array from directions  $\theta_1, \theta_2, \dots, \theta_K$ . At any instant these  $K$  signals  $u_1(t), u_2(t), \dots, u_K(t)$  are phase-delayed, amplitude-weighted replicas of one of them — say, the first — and hence

$$u_k(t) = \alpha_k u_1(t), \quad k = 1, 2, \dots, K$$

where  $\alpha_k$  represents the complex attenuation of the  $k^{\text{th}}$  signal with respect to the first signal,  $u_1(t)$ . Using complex signal representation, the received signal at the  $i^{\text{th}}$  sensor can be expressed as

(1) The symbol  $[x]$  stands for the integer part of  $x$ .

$$x(t) = \sum_{k=1}^M x_k(t) \exp(-j2\pi f_k t) + n(t) = \cos \theta_k + n(t) \quad (2)$$

where the unknown distance is taken to be half wavelength and  $n(t)$  represents the additive noise in the  $M$  sensor. It is assumed that the signals and noise are stationary zero mean uncorrelated random processes, and further the noises are assumed to be uncorrelated and identical between channels with common variance  $\sigma^2$ . Knowing  $\theta_k$  is common across all  $M$  channels and with  $\omega_k = 2\pi f_k$ ,  $k = 1, 2, \dots, M$ , we have

$$x = \frac{1}{M} \sum_{k=1}^M x_k = \frac{1}{M} \sum_{k=1}^M \cos \theta_k + \frac{1}{M} \sum_{k=1}^M n_k = \frac{1}{M} \sum_{k=1}^M \cos \theta_k + n \quad (3)$$

where  $n = \frac{1}{M} \sum_{k=1}^M n_k$  is zero mean white flat noise and

$$x_k = \frac{1}{M} \sum_{i=1}^M x_{ki} = \frac{1}{M} \sum_{i=1}^M \cos \theta_k + \frac{1}{M} \sum_{i=1}^M n_{ki} \quad (4)$$

$$x_k = \frac{1}{M} \sum_{i=1}^M x_{ki} = \frac{1}{M} \sum_{i=1}^M \cos \theta_k + \frac{1}{M} \sum_{i=1}^M n_{ki} \quad (5)$$

$$x = \frac{1}{M} \sum_{k=1}^M x_k = \frac{1}{M} \sum_{k=1}^M \cos \theta_k + \frac{1}{M} \sum_{k=1}^M \sum_{i=1}^M n_{ki} \quad (6)$$

with  $n_{ki}$  representing the flat white noise associated with the antenna angle  $\theta_k$  in  $i$ th

$$n_{ki} = \frac{1}{M} \sum_{j=1}^M \exp(-j2\pi f_k t) n_{ij} = \frac{1}{M} \sum_{j=1}^M \exp(-j2\pi f_k t) n_{ij} = \frac{1}{M} \sum_{j=1}^M \exp(-j2\pi f_k t) n_{ij} \quad (7)$$

Here  $\mathbf{x}$  is an  $M \times 1$  column vector with  $M$  uncorrelated components and  $\mathbf{x}^T \mathbf{x} = M$  and hence  $\mathbf{x}$  is of rank  $M$ . From our assumption it now follows that the input-output covariance matrix  $\mathbf{R} = E[\mathbf{x} \mathbf{x}^T]$  has the form

$$\mathbf{R} = \mathbf{A} \mathbf{R}_1 \mathbf{A}^T + \sigma^2 \mathbf{I} \quad (8)$$

where  $\mathbf{R}_1 = E[\mathbf{x}_k \mathbf{x}_k^T]$  represents the source covariance matrix that remains nonsingular so long as the sources are at most partially correlated. In that case,

2) How many  $\lambda$ 's does the covariance matrix possess

$\mathbf{A}\mathbf{B}_s\mathbf{A}^T$  is also of rank  $M$  and hence  $\lambda_1 = \lambda_2 = \dots = \lambda_M = 0$  and  $\{\beta_1, \beta_2, \dots, \beta_M\}$  are the eigenvalues and the corresponding eigenvectors of  $\mathbf{B}$ . Then the above rank property implies that  $\lambda_1 = \lambda_2^2 = \dots = \lambda_M = 0$  and further [8]

$$\beta^T \mathbf{A} \mathbf{a}_s = 0, \quad s = 1, 2, \dots, M, \quad \mathbf{M}^T \mathbf{A} = \mathbf{0}^T, \quad \mathbf{A}^T \mathbf{A} = \mathbf{I} \quad (9)$$

The high-resolution eigenstructure-based techniques make use of (9) (these relationships are obtained when  $\mathbf{B}_s$  is of full rank) to estimate the actual directions of arrival  $\theta_1, \dots, \theta_M$  respectively. However, when the signals are coherent as in (1), the above condition is no longer true and different relations hold. In that case using (1) in (4) and with  $\theta = \theta_1 = \dots = \theta_M = \theta$  it can be seen that

$$\mathbf{B}_s = \mathbf{A} \mathbf{a} \mathbf{a}^T, \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{j\theta} \\ \alpha_2 e^{j\theta} \\ \vdots \\ \alpha_M e^{j\theta} \end{bmatrix} \quad (10)$$

and from (4) the sensor output covariance matrix reduces to

$$\mathbf{R} = \mathbf{A} \mathbf{a} \mathbf{a}^T \mathbf{A}^T = \sigma^2 \mathbf{I} \triangleq \mathbf{b} \mathbf{b}^T + \sigma^2 \mathbf{I} \quad (11)$$

Here  $\mathbf{b} = \mathbf{A} \mathbf{a}$  and again reasoning as before, it follows that  $\lambda_1 = \lambda_2 = \dots = \lambda_M = \sigma^2$  and hence

$$\beta^T \mathbf{b} = 0, \quad \beta = 2, 3, \dots, M \quad (12)$$

Because of their Vandermonde structure, no linear combination of direction vectors can result in another direction vector. Consequently  $\mathbf{b}$  is no longer a legitimate direction vector and hence (12) will not be able to estimate any true arrival angles. The crucial role played by the nonsingularity of  $\mathbf{R}_u$  in this discussion has prompted Evans *et al.* and subsequently Shan *et al.* to introduce a preprocessing scheme [9]–[11] which guarantees full rank for the equivalent  $\mathbf{R}_u$  in (8) even when the signals are all coherent. This preprocessing spatial smoothing scheme starts by dividing a uniform linear array with  $M_u$  sensors into uniformly overlapping subarrays of size  $M$  (see Fig. 1). Let  $\mathbf{x}_l^T(t)$  stand for the output of the  $l^{\text{th}}$  subarray for  $l = 1, 2, \dots, L \triangleq$

$M_0 - M + 1$ , where  $L$  denotes the total number of these forward subarrays. Using (2) - (6) we have

$$\mathbf{x}_l^f(t) \triangleq [x_l(t), x_{l+1}(t), \dots, x_{l+M-1}(t)]^T = \mathbf{A} \mathbf{B}^{l-1} \mathbf{u}(t) + \mathbf{n}_l(t), \quad 1 \leq l \leq L \quad (13)$$

where  $\mathbf{B}^{l-1}$  denotes the  $(l-1)^{\text{th}}$  power of the  $K \times K$  diagonal matrix

$$\mathbf{B} = \text{diag} [\nu_1, \nu_2, \dots, \nu_K] \quad ; \quad \nu_i = \exp(-j\omega_i) \quad , \quad i = 1, 2, \dots, K. \quad (14)$$

Then, the covariance matrix of the  $l^{\text{th}}$  subarray is given by

$$\mathbf{R}_l^f = E [\mathbf{x}_l^f(t) (\mathbf{x}_l^f(t))^H] = \mathbf{A} \mathbf{B}^{l-1} \mathbf{R}_u (\mathbf{B}^{l-1})^H \mathbf{A}^H + \sigma^2 \mathbf{I}. \quad (15)$$

Following [9] - [11] define the forward spatially smoothed covariance matrix,  $\mathbf{R}^f$  as the sample mean of the forward subarray covariance matrices and this gives

$$\mathbf{R}^f = \frac{1}{L} \sum_{l=1}^L \mathbf{R}_l^f = \mathbf{A} \mathbf{R}_u^f \mathbf{A}^H + \sigma^2 \mathbf{I}. \quad (16)$$

In a completely coherent environment, using (10) the forward-smoothed source covariance matrix  $\mathbf{R}_u^f$  takes the form

$$\mathbf{R}_u^f \triangleq \frac{1}{L} \sum_{l=1}^L \mathbf{B}^{l-1} \mathbf{R}_u \mathbf{B}^{l-1H} = \frac{1}{L} \mathbf{C} \mathbf{C}^H \quad (17)$$

where

$$\mathbf{C} = \begin{bmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_K \\ \sigma_1 \nu_1 & \sigma_2 \nu_2 & \dots & \sigma_K \nu_K \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1 \nu_1^{L-1} & \sigma_2 \nu_2^{L-1} & \dots & \sigma_K \nu_K^{L-1} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_K \\ \sigma_1 \nu_1 & \sigma_2 \nu_2 & \dots & \sigma_K \nu_K \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1 \nu_1^{L-1} & \sigma_2 \nu_2^{L-1} & \dots & \sigma_K \nu_K^{L-1} \end{bmatrix} \triangleq \mathbf{D} \mathbf{V},$$

Clearly the rank of  $\mathbf{R}_u^f$  is equal to the rank of  $\mathbf{C}$ . Since  $\mathbf{C} = \mathbf{D} \mathbf{V}$  and the square matrix  $\mathbf{D}$  is of full rank, the rank of  $\mathbf{C}$  is the same as that of  $\mathbf{V}$ . Now the rank of the  $K \times L$  Vandermonde matrix  $\mathbf{V}$  is  $\text{rank}(\mathbf{V}) = \min(K, L)$  and hence  $\text{rank}(\mathbf{R}_u^f) = \min(K, L) \leq K$ .

Thus, if  $L = M_o - M + 1 \geq K$  or equivalently  $M_o \geq M + K - 1$ , the smoothed source covariance matrix  $\mathbf{R}_u^f$  is nonsingular and  $\mathbf{R}^f$  has exactly the same form as the covariance matrix for a noncoherent case. Therefore the conclusions in (9) will hold good for  $\mathbf{R}^f$  in (16), and as pointed out by Shan *et al.*, one can successfully apply the eigenstructure methods to this smoothed covariance matrix regardless of the coherence of the signals. However, in this case, the number of sensor elements  $M_o$  must be at least  $(M + K - 1)$ , and recalling from (9) that the size  $M$  of each subarray must also be at least  $K + 1$ , it follows that the minimum number of sensors needed is  $2K$  compared to  $K + 1$  for the conventional one. In what follows we present the improved spatial smoothing scheme that makes use of the forward and appropriate backward subarrays to reduce the required number of sensor elements to  $[3K/2]$ .

Towards this purpose, additional  $L$  backward subarrays are generated from the same set of sensors by grouping elements at  $\{M_o, M_o - 1, \dots, M_o - M + 1\}$  to form the first backward subarray and elements at  $\{M_o - 1, M_o - 2, \dots, M_o - M\}$  to form the second one, etc. (see Fig. 1). Let  $\mathbf{x}_l^b(t)$  denote the complex conjugate of the output of the  $l^{\text{th}}$  backward subarray for  $l = 1, 2, \dots, L$ , where  $L$  as before denotes the total number  $(M_o - M + 1)$  of these subarrays. Thus

$$\mathbf{x}_l^b(t) = [\mathbf{x}_{M_o - l + 1}^*(t), \mathbf{x}_{M_o - l}^*(t), \dots, \mathbf{x}_{M_o - l + 1}^*(t)]^T = \mathbf{A} \mathbf{B}^{l-1} \left( \mathbf{B}^{M_o-1} \mathbf{u}(t) \right)^* + \tilde{\mathbf{n}}_l^*(t), \quad 1 \leq l \leq L \quad (19)$$

where  $\mathbf{B}$  is as defined in (14). The covariance matrix of the  $l^{\text{th}}$  backward subarray is given by

$$\mathbf{R}^b = E[\mathbf{x}_l^b(t) (\mathbf{x}_l^b(t))^H] = \mathbf{A} \mathbf{B}^{l-1} \mathbf{R}_u^* (\mathbf{B}^{l-1})^H \mathbf{A}^H + \sigma^2 \mathbf{I} \quad (20)$$

with

$$\mathbf{R}_u^* \triangleq \mathbf{B}^{-(M_o-1)} E[\mathbf{u}^*(t) \mathbf{u}^T(t)] (\mathbf{B}^{-(M_o-1)})^H = \mathbf{B}^{-(M_o-1)} \mathbf{R}_u^* (\mathbf{B}^{-(M_o-1)})^H \quad (21)$$

As before, define the spatially smoothed backward subarray covariance matrix  $\mathbf{R}^b$  as the sample mean of these subarray covariance matrices, i.e.,



$$\mathbf{R}^b = \frac{1}{L} \sum_{l=1}^L \mathbf{R}_l^b = \mathbf{A} \mathbf{R}_u^b \mathbf{A}^\dagger + \sigma^2 \mathbf{I}. \quad (22)$$

In a completely coherent environment  $\mathbf{R}_u$  is given by (10) and in that case using (10) in (21)  $\mathbf{R}_u$  simplifies to

$$\mathbf{R}_u = \boldsymbol{\delta} \boldsymbol{\delta}^\dagger, \quad (23)$$

where

$$\boldsymbol{\delta} = [\delta_1, \delta_2, \dots, \delta_K]^T; \quad \delta_k = \alpha_k^* \nu_k^{-(M_o-1)}, \quad k = 1, 2, \dots, K \quad (24)$$

with  $\nu_k$ ,  $k = 1, 2, \dots, K$  as defined in (14). Finally, using (23) the backward-smoothed source covariance matrix  $\mathbf{R}_u^b$  is given by

$$\mathbf{R}_u^b \triangleq \frac{1}{L} \sum_{l=1}^L \mathbf{B}^{l-1} \mathbf{R}_u (\mathbf{B}^{l-1})^\dagger = \frac{1}{L} \mathbf{E} \mathbf{E}^\dagger \quad (25)$$

where

$$\mathbf{E} = [\boldsymbol{\delta}, \mathbf{B} \boldsymbol{\delta}, \mathbf{B}^2 \boldsymbol{\delta}, \dots, \mathbf{B}^{L-1} \boldsymbol{\delta}] = \mathbf{F} \mathbf{V} \quad (26)$$

with  $\mathbf{V}$  as in (18) and

$$\mathbf{F} = \text{diag} [\delta_1, \delta_2, \dots, \delta_K]. \quad (27)$$

Reasoning as before it is easy to see that the backward spatially smoothed covariance matrix  $\mathbf{R}^b$  will be of full rank so long as  $\mathbf{R}_u^b$  is nonsingular, and this is guaranteed whenever  $L \geq K$ . Again, it follows that the backward subarray averaging scheme also requires at most  $2K$  sensor elements to estimate the directions of arrival of  $K$  sources irrespective of their coherence.

It remains to show that by simultaneous use of the forward and backward subarray averaging schemes, it is possible to further reduce the number of extra sensor elements. To see this, following Evans *et al.* [10], define the forward/backward smoothed covariance matrix  $\hat{\mathbf{R}}$  as the mean of  $\mathbf{R}^f$  and  $\mathbf{R}^b$ ; i.e.

$$\tilde{\mathbf{R}} = \frac{\mathbf{R}^f + \mathbf{R}^b}{2}. \quad (28)$$

Using (16), (17), (22) and (25) in (28) we have

$$\tilde{\mathbf{R}} = \mathbf{A} \left[ \frac{1}{2L} (\mathbf{C} \mathbf{C}^\dagger + \mathbf{E} \mathbf{E}^\dagger) \right] \mathbf{A}^\dagger + \sigma^2 \mathbf{I} = \mathbf{A} \tilde{\mathbf{R}}_u \mathbf{A}^\dagger + \sigma^2 \mathbf{I} \quad (29)$$

with

$$\tilde{\mathbf{R}}_u = \frac{1}{2L} [\mathbf{C} \mathbf{C}^\dagger + \mathbf{E} \mathbf{E}^\dagger] = \frac{1}{2L} \mathbf{G} \mathbf{G}^\dagger. \quad (30)$$

Here

$$\begin{aligned} \mathbf{G} &= [\alpha, \mathbf{B} \alpha, \mathbf{B}^2 \alpha, \dots, \mathbf{B}^{L-1} \alpha, \delta, \mathbf{B} \delta, \mathbf{B}^2 \delta, \dots, \mathbf{B}^{L-1} \delta] \\ &= [\mathbf{D} \mathbf{V} \mid \mathbf{F} \mathbf{V}] = \mathbf{D} [\mathbf{V} \mid \mathbf{H} \mathbf{V}] \triangleq \mathbf{D} \mathbf{G}_0, \end{aligned} \quad (31)$$

with  $\mathbf{D}$ ,  $\mathbf{V}$  as in (18) and

$$\mathbf{H} = \text{diag} [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_K] ; \quad \varepsilon_k = \delta_k / \alpha_k, \quad k = 1, 2, \dots, K. \quad (32)$$

We will now prove that the modified source covariance matrix  $\tilde{\mathbf{R}}_u$  given by (30) will be nonsingular regardless of the coherence of the  $K$  signal sources so long as  $2L \geq K$ , provided that whenever equality holds among some of the members of the set  $\{\varepsilon_k\}_{k=1}^K$  in (32), the largest subset with equal entries must at most be of size  $L$ .

To appreciate this restriction, first consider the case where all  $\varepsilon_k$ ,  $k = 1, 2, \dots, K$  are equal. In that case it is easy to see that  $\mathbf{G}_0$  and hence  $\tilde{\mathbf{R}}_u$  will be of rank  $\min(L, K)$  irrespective of the backward smoothing. However, in practice this equality condition almost never occurs. This is because  $\alpha_k$  in (1), which represents the complex attenuation of the  $k^{\text{th}}$  source with respect to the reference source, is a signal property, and  $\delta_k$  in (24), which is a function of the interelement phase delay of the  $k^{\text{th}}$  source with respect to the reference element, is mainly an array geometry property.

Thus, in an actual situation all  $e_k, k = 1, 2, \dots, K$  will be distinct and the simultaneous equality condition for all of them makes it an almost never occurring event. From these arguments it also follows that the above restrictions on the equality among some of the  $e_k$ 's will almost always be satisfied. To be specific with regard to these restrictions, we will assume that

$$e_i \neq e_j, \text{ for any } i = 1, 2, \dots, L, \text{ and } j = L+1, L+2, \dots, K. \quad (33)$$

A special case of the general situation, where all  $\alpha_k, k = 1, 2, \dots, K$ , in (1) are real, is treated in [13]. In that case using (24) and (32) in (31) it is easy to see that  $G_0$  is a Vandermonde matrix with distinct columns and hence is of rank  $K$  so long as  $2L \geq K$ . This, however, is an unrealistic assumption, as in practice, all  $\alpha_k$ 's will be invariably complex numbers and in that case it is necessary to argue differently as follows.

From (30),  $\hat{R}_u$  will be nonsingular so long as  $G$  is of full row rank, and using (31) this is further equivalent to having full row rank for  $G_0$ . Clearly, for  $G$  (or  $G_0$ ) to have full row rank it is necessary that  $2L \geq K$  and with  $L = M_o - M + 1$ , this reduces to  $2M_o \geq 2M + K - 2$ . Again recalling that in the presence of  $K$  signals the size  $M$  of each subarray must be at least  $K + 1$ , it follows that the number of sensors  $M_o$  needed must satisfy  $2M_o \geq 3K$  or, equivalently, the minimum number of sensors must be at least  $\lceil 3K/2 \rceil$ . To see that this requirement is also sufficient, consider the quadratic product

$$y^\dagger G_0 G_0^\dagger y = y^\dagger V V^\dagger y + y^\dagger H V V^\dagger H^\dagger y \quad (34)$$

where  $y$  is any arbitrary  $K \times 1$  vector. We will show that

$$y^\dagger G_0 G_0^\dagger y > 0 \quad (35)$$

for any  $y \neq 0$ , thus proving the positive-definite property of  $G_0 G_0^\dagger$  or  $\hat{R}_u$ . Clearly (35) needs to be demonstrated only for a typical  $y_0 \in N(V^\dagger)$ , the null space of  $V^\dagger$ . In that case  $V^\dagger y_0 = 0$  and hence the first term in (34) reduces to zero. To prove our claim, it

is enough to show that for such a typical  $y_0$ ,  $H^\dagger y_0$  does not belong to  $N(V^\dagger)$ . Since the Vandermonde structured matrix  $V^\dagger$  is of full row rank  $L$ , the dimension of  $N(V^\dagger)$  is  $K-L$ . Let  $v_{L+1}, v_{L+2}, \dots, v_K$  be a set of linearly independent basis vectors for  $N(V^\dagger)$ . With respect to the basis vectors for the  $K$ -dimensional space, these null space basis vectors can always be chosen such that [14],

$$v_l = [v_{1l}, v_{2l}, \dots, v_{Ll}, 0, \dots, 0, 1, 0, \dots, 0]^T. \quad (36)$$

(In (36) the 1 is at the  $l^{th}$  location.) These  $v_l, l = L+1, L+2, \dots, K$  are linearly independent and, moreover, for any  $j \in \{L+1, L+2, \dots, K\}$ , using the diagonal nature of  $H$  it is also easy to see that  $H^\dagger v_j$  is linearly independent of the remaining  $v_l, l = L+1, \dots, K, l \neq j$ . Further the pair  $v_j$  and  $H^\dagger v_j, j = L+1, L+2, \dots, K$ , is also linearly independent of each other. To see this note that because of the full row rank property of  $V^\dagger$ , at least one of the  $v_{il}, i = 1, 2, \dots, L$  in (36) must be nonzero for every  $l$ . Let  $v_{i_o j}$  be such an entry in  $v_j$ . Then the minor formed by the  $i_o^{th}$  and  $j^{th}$  rows of the matrix  $[v_j | H^\dagger v_j]$  has the form

$$\begin{vmatrix} v_{i_o j} & \epsilon_{i_o}^* v_{i_o j} \\ 1 & \epsilon_j^* \end{vmatrix} = v_{i_o j} (\epsilon_j^* - \epsilon_{i_o}^*) \quad (37)$$

and is nonzero from (33). Thus the matrix  $[v_j | H^\dagger v_j]$  is of rank 2. This proves the linear independence of  $v_j$  and  $H^\dagger v_j$ . From the above discussion it follows that  $H^\dagger v_j$  is linearly independent of  $v_j, j = L+1, L+2, \dots, K$  and hence  $H^\dagger v_j \notin N(V^\dagger), j = L+1, L+2, \dots, K$ . Now for any  $y_0 \in N(V^\dagger)$  we have

$$y_0 = \sum_{j=L+1}^K k_j v_j, \quad (38)$$

which gives

$$H^\dagger y_0 = \sum_{j=L+1}^K k_j H^\dagger v_j. \quad (39)$$

Since all  $k_j$  cannot be zero in (39), it follows that  $\mathbf{H}^\dagger \mathbf{y}_0 \notin \mathbf{N}(\mathbf{V}^\dagger)$  and hence  $\mathbf{V}^\dagger \mathbf{H}^\dagger \mathbf{y}_0 \neq \mathbf{O}$ . This proves our claim and establishes that  $\tilde{\mathbf{R}}_\mu$  will be nonsingular under the mild restrictions in (33). In that case the eigenvalues of  $\tilde{\mathbf{R}}$  satisfy  $\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \dots \geq \bar{\lambda}_K > \bar{\lambda}_{K+1} = \bar{\lambda}_{K+2} = \dots = \bar{\lambda}_M = \sigma^2$ . Consequently, as in (9), the eigenvectors corresponding to equal eigenvalues are orthogonal to the direction vectors associated with the true directions of arrival; i.e.

$$\tilde{\beta}_i^\dagger \mathbf{a}(\omega_k) = 0, \quad i = K+1, K+2, \dots, M, \quad k = 1, 2, \dots, K \quad (40)$$

Here  $\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_M$  are the eigenvectors of  $\tilde{\mathbf{R}}$  corresponding to the eigenvalues  $\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_M$ , respectively.

To summarize, we have proved that so long as the number of sensor elements is at least  $\lceil 3K/2 \rceil$ , (with  $K$  representing the number of signal sources present in the scene), it is almost always possible to estimate all arrival angles irrespective of the signal correlations by simultaneous use of the forward and backward subarray averaging scheme. Since the smoothed covariance matrix  $\tilde{\mathbf{R}}$  in (28) has exactly the same form as the covariance matrix for some noncoherent situation as in (8), the eigenstructure-based techniques can be applied to this smoothed covariance matrix, irrespective of the coherence of the signals, to successfully estimate their directions of arrival.

The Appendix extends the proof for the forward/backward smoothing scheme to a mixed source scene consisting of partially correlated signals with complete coherence among some of them.

### 2.1.3 Simulation Results

In this section simulation results are presented to illustrate the performance of the forward/backward spatial smoothing scheme and to compare it with the conventional eigenstructure-based techniques [3].

Fig. 2 represents a coherent source scene where the reference signal arriving from  $70^\circ$  undergoes multipath reflection, resulting in three additional coherent arrivals

along  $45^\circ$ ,  $115^\circ$  and  $127^\circ$ . A six-element uniform array is used to receive these signals. The input signal-to-noise ratio (SNR) of the reference signal is 5 dB, and the attenuation coefficients of the three coherent sources are taken to be  $(0.4, 0.8)$ ,  $(-0.3, -0.7)$  and  $(0.5, -0.6)$ , respectively. In the notation  $\alpha = (a, b)$ , here  $a$  and  $b$  represent the real and imaginary parts, respectively, of the complex attenuation coefficient  $\alpha$ . Three hundred data samples are used to estimate the array output covariance matrix using the standard maximum likelihood procedure. The application of the conventional eigenstructure method [3] to this covariance matrix resulted in Fig. 2.a. However, first applying the forward/backward spatial smoothing scheme with two forward and two backward ( $L=2$ ) subarrays of five ( $M=5$ ) sensors each, and then reapplying the eigenstructure technique on the smoothed covariance matrix  $\hat{\mathbf{R}}$  resulted in Fig. 2.b. All four directions of arrival can be clearly identified and the improvement in performance in terms of resolvability, irrespective of the signal coherence, is also visible in this case.

#### 2.1.4 Conclusions

This report reexamines the problem of locating the directions of arrival of coherent signals and in that context a spatial smoothing scheme, first introduced by Evans *et al.* and analyzed by Shan *et al.*, is further investigated. It is proved here that by simultaneous use of a set of forward and complex conjugated backward subarrays, it is always possible to estimate any  $K$  directions of arrival using at most  $\lceil 3K/2 \rceil$  sensor elements. This is made possible by creating a smoothed array output covariance matrix that is structurally identical to a covariance matrix in some noncoherent situation, thus enabling one to correctly identify all directions of arrival by incorporating the eigenstructure-based techniques [3] on this smoothed matrix. This is a considerable saving compared to the forward-only smoothing scheme [11] that requires as many extra sensor elements as the total number of coherent signals present in the scene.

## Appendix

### Coherent and Correlated Signal Scene

We will demonstrate here that the forward/backward smoothing scheme discussed in section 2.1.2 readily extends to the general situation where the source scene consists of  $K + J$  signals  $u_1(t), u_2(t), \dots, u_K(t), u_{K+1}(t), \dots, u_{K+J}(t)$ , of which the first  $K$  signals are completely coherent and the last  $(J + 1)$  signals are partially correlated. Thus the coherent signals are partially correlated with the remaining set of signals. Further, the respective arrival angles are assumed to be  $\theta_1, \theta_2, \dots, \theta_K, \theta_{K+1}, \dots, \theta_{K+J}$ . As before, the signals are taken to be uncorrelated with the noise and the noise is assumed to be identical and uncorrelated from element to element. With symbols as defined in the text and using (2), the output  $x_i(t)$  of the  $i^{th}$  sensor element at time  $t$  in this case can be written as

$$x_i(t) = u_1(t) \sum_{k=1}^K \alpha_k \exp(-j(i-1)\omega_k) + \sum_{k=1}^{K+J} u_k(t) \exp(-j(i-1)\omega_k) + n_i(t),$$

$$i = 1, 2, \dots, M. \quad (A.1)$$

With  $\mathbf{x}(t)$  as in (3), this gives

$$\mathbf{x}(t) = \tilde{\mathbf{A}} \mathbf{v}(t) + \mathbf{n}(t), \quad (A.2)$$

where

$$\tilde{\mathbf{A}} = \sqrt{M} \left[ \mathbf{a}(\omega_1), \mathbf{a}(\omega_2), \dots, \mathbf{a}(\omega_K), \mathbf{a}(\omega_{K+1}), \dots, \mathbf{a}(\omega_{K+J}) \right] \quad (A.3)$$

with  $\mathbf{a}(\omega_k); k = 1, 2, \dots, K + J$  as defined in (7) and

$$\mathbf{v}(t) \triangleq \left[ \mathbf{u}_1(t) \mid \mathbf{u}_2(t) \right]. \quad (A.4)$$

Here

$$\mathbf{u}_1(t) = \left[ u_1(t), u_2(t), \dots, u_K(t) \right]^T = u_1(t) \boldsymbol{\alpha} \quad (A.5)$$

with  $\alpha$  as in (10) and

$$\mathbf{u}_2(t) = \left[ u_{K+1}(t), u_{K+2}(t), \dots, u_{K+J}(t) \right]^T. \quad (\text{A.6})$$

Following (13)-(17), (19)-(20), (25) and (28), the forward/backward smoothed covariance matrix  $\tilde{\mathbf{R}}$  in this case can be written as

$$\tilde{\mathbf{R}} = \tilde{\mathbf{A}} \tilde{\mathbf{R}}_0 \tilde{\mathbf{A}}^\dagger + \sigma^2 \mathbf{I}, \quad (\text{A.7})$$

where

$$\tilde{\mathbf{R}}_0 = \frac{1}{2L} \sum_{l=1}^L \tilde{\mathbf{B}}'^{-1} (\mathbf{R}_v + \mathbf{R}_v^\dagger) (\tilde{\mathbf{B}}'^{-1})^\dagger. \quad (\text{A.8})$$

It remains to show that  $\tilde{\mathbf{R}}_0$  is of full rank irrespective of the coherency among some of the arrivals. Here

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_2 \end{bmatrix} \quad (\text{A.9})$$

where

$$\mathbf{B}_1 = \text{diag} [\nu_1, \nu_2, \dots, \nu_K] \quad (\text{A.10})$$

and

$$\mathbf{B}_2 = \text{diag} [\nu_{K+1}, \nu_{K+2}, \dots, \nu_{K+J}] \quad (\text{A.11})$$

with  $\nu_k, k = 1, \dots, K+J$  as given by (14) and

$$\mathbf{R}_v \triangleq E[\mathbf{v}(t) \mathbf{v}^\dagger(t)] = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{12}^\dagger & \mathbf{R}_{22} \end{bmatrix}. \quad (\text{A.12})$$

Using (A.4) - (A.6), it is easy to see that

$$\mathbf{R}_{11} = E[\mathbf{u}_1(t) \mathbf{u}_1^\dagger(t)] = \alpha \alpha^\dagger \quad (\text{A.13})$$

where  $\alpha$  is as before and  $E[|u_1(t)|^2] = 1$ . Similarly

$$\mathbf{R}_{12} = E[\mathbf{u}_1(t) \mathbf{u}_2^\dagger(t)] = \alpha \gamma^\dagger \quad (\text{A.14})$$

with



$$\gamma = [\gamma_1, \gamma_2, \dots, \gamma_J]^T, \quad (\text{A.15})$$

where

$$\gamma_i \triangleq E[u_1(t)u_{K+i}^*(t)], \quad i = 1, 2, \dots, J, \quad (\text{A.16})$$

and

$$\mathbf{R}_{22} = E[\mathbf{u}_2(t)\mathbf{u}_2^\dagger(t)]. \quad (\text{A.17})$$

From the partially correlated assumption among the later  $J$  signals, it follows that their correlation matrix  $\mathbf{R}_{22}$  is of full rank and hence it has the representation

$$\mathbf{R}_{22} = \Lambda \Lambda^\dagger \quad (\text{A.18})$$

where  $\Lambda$  is again a full rank matrix of size  $J \times J$ . In a similar manner following (21),  $\mathbf{R}_v$  can be written as

$$\mathbf{R}_v = \tilde{\mathbf{B}}^{-(M_o-1)} \mathbf{R}_v^* \left( \tilde{\mathbf{B}}^{-(M_o-1)} \right)^\dagger = \begin{bmatrix} \tilde{\mathbf{R}}_{11} & \tilde{\mathbf{R}}_{12} \\ \tilde{\mathbf{R}}_{12}^\dagger & \tilde{\mathbf{R}}_{22} \end{bmatrix}, \quad (\text{A.19})$$

and proceeding as before,

$$\tilde{\mathbf{R}}_{11} = \delta \delta^\dagger \quad (\text{A.20})$$

with  $\delta$  as in (24) and

$$\tilde{\mathbf{R}}_{12} = \delta \tilde{\gamma}^\dagger \quad (\text{A.21})$$

with

$$\tilde{\gamma} = [\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_J]^T, \quad (\text{A.22})$$

where

$$\tilde{\gamma}_i = \gamma_i^* \nu_{K+i}^{-(M_o-1)}, \quad i = 1, 2, \dots, J. \quad (\text{A.23})$$

Here  $\gamma_i$  is as defined in (A.16) and  $\nu_{K+i}$  is obtained by extending the definition in (14). Further,

$$\tilde{\mathbf{R}}_{22} = \mathbf{B}_2^{-(M_o-1)} \mathbf{R}_{22} (\mathbf{B}_2^{-(M_o-1)})^\dagger = \tilde{\Lambda} \tilde{\Lambda}^\dagger \quad (\text{A.24})$$

with

$$\tilde{\mathbf{A}} = \mathbf{B}_j^{-1} \mathbf{M}_j^{-1} \mathbf{A} \quad (\text{A.25})$$

where  $\tilde{\mathbf{A}}$  again is a full rank matrix of size  $J \times J$ . With (A.9) - (A.25) in (A.8), it simplifies to

$$\begin{aligned} \hat{\mathbf{R}}_0 &= \frac{1}{2L} \begin{bmatrix} \sum_{j=1}^L \mathbf{B}_j^{-1} (\mathbf{R}_{1j} + \hat{\mathbf{R}}_{1j}) (\mathbf{B}_j^{-1})^T & \sum_{j=1}^L \mathbf{B}_j^{-1} (\mathbf{R}_{1j} + \hat{\mathbf{R}}_{1j}) (\mathbf{B}_j^{-1})^T \\ \sum_{j=1}^L \mathbf{B}_j'^{-1} (\mathbf{R}_{2j} + \hat{\mathbf{R}}_{2j}) (\mathbf{B}_j'^{-1})^T & \sum_{j=1}^L \mathbf{B}_j'^{-1} (\mathbf{R}_{2j} + \hat{\mathbf{R}}_{2j}) (\mathbf{B}_j'^{-1})^T \end{bmatrix} \\ &= \frac{1}{2L} \begin{bmatrix} \mathbf{G}_1 \mathbf{G}_1^T & \mathbf{G}_1 \mathbf{G}_2^T \\ \mathbf{G}_2^T \mathbf{G}_1 & \mathbf{G}_2 \mathbf{G}_2^T \end{bmatrix} = \frac{1}{2L} \begin{bmatrix} \mathbf{G}_1 & \mathbf{O} \\ \mathbf{G}_2 & \mathbf{G}_4 \end{bmatrix} \begin{bmatrix} \mathbf{G}_1^T & \mathbf{G}_2^T \\ \mathbf{O} & \mathbf{G}_4^T \end{bmatrix} \end{aligned} \quad (\text{A.26})$$

where

$$\mathbf{G}_1 = [\boldsymbol{\alpha}, \mathbf{B}_1 \boldsymbol{\alpha}, \dots, \mathbf{B}_1^{L-1} \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{B}_1 \boldsymbol{\delta}, \dots, \mathbf{B}_1^{L-1} \boldsymbol{\delta}], \quad (\text{A.27})$$

$$\mathbf{G}_2 = [\boldsymbol{\gamma}, \mathbf{B}_2 \boldsymbol{\gamma}, \dots, \mathbf{B}_2^{L-1} \boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}}, \mathbf{B}_2 \tilde{\boldsymbol{\gamma}}, \dots, \mathbf{B}_2^{L-1} \tilde{\boldsymbol{\gamma}}], \quad (\text{A.28})$$

$$\mathbf{G}_3 = [\boldsymbol{\Lambda}, \mathbf{B}_2 \boldsymbol{\Lambda}, \dots, \mathbf{B}_2^{L-1} \boldsymbol{\Lambda}, \tilde{\boldsymbol{\Lambda}}, \mathbf{B}_2 \tilde{\boldsymbol{\Lambda}}, \dots, \mathbf{B}_2^{L-1} \tilde{\boldsymbol{\Lambda}}], \quad (\text{A.29})$$

and  $\mathbf{G}_4$  satisfies

$$\mathbf{G}_3 \mathbf{G}_3^T = \mathbf{G}_2 \mathbf{G}_2^T + \mathbf{G}_4 \mathbf{G}_4^T. \quad (\text{A.30})$$

Define

$$\hat{\mathbf{G}} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{O} \\ \mathbf{G}_2 & \mathbf{G}_4 \end{bmatrix}. \quad (\text{A.31})$$

Then

$$\hat{\mathbf{R}}_0 = \frac{1}{2L} \hat{\mathbf{G}} \hat{\mathbf{G}}^\dagger. \quad (\text{A.32})$$

Clearly the rank of  $\hat{\mathbf{R}}_0$  is the same as that of  $\hat{\mathbf{G}}$ . An examination of (A.27) shows that  $\mathbf{G}_1 \mathbf{G}_1^\dagger$  is the average of the source covariance matrix corresponding to the completely coherent situation (see (31)) and hence from the result derived in section 2.1.2, it follows that  $\mathbf{G}_1 \mathbf{G}_1^\dagger$  is of full rank  $K$  so long as  $L \geq [K/2]$ . Now it remains to show that  $\mathbf{G}_4$  is also of full row rank  $J$ , which together with (A.31) implies that  $\hat{\mathbf{G}}$  and hence  $\hat{\mathbf{R}}_0$  is of full rank  $K + J$ . From (A.28) - (A.30) we have

$$\begin{aligned} \mathbf{G}_4 \mathbf{G}_4^\dagger &= \mathbf{G}_3 \mathbf{G}_3^\dagger - \mathbf{G}_2 \mathbf{G}_2^\dagger \\ &= \left[ \sum_{l=1}^L \mathbf{B}_2'^{-1} (\Lambda \Lambda^\dagger - \gamma \gamma^\dagger) (\mathbf{B}_2'^{-1})^\dagger + \sum_{l=1}^L \mathbf{B}_2'^{-1} (\bar{\Lambda} \bar{\Lambda}^\dagger - \bar{\gamma} \bar{\gamma}^\dagger) (\mathbf{B}_2'^{-1})^\dagger \right] \end{aligned} \quad (\text{A.33})$$

In the first summation here,  $\Lambda$  and  $\gamma$  are matrices of ranks  $J$  and 1 respectively and hence the matrix  $(\Lambda \Lambda^\dagger - \gamma \gamma^\dagger)$  is at least of rank  $J - 1$ . Once again, resorting to the argument used in establishing (35) in section 2.1.2, it follows that each summation and hence  $\mathbf{G}_4$  is of full row rank  $J$  so long as  $L > 1$ . This establishes the nonsingularity of  $\hat{\mathbf{R}}_0$  for  $L \geq [K/2]$ . As a result, the smoothed covariance matrix  $\hat{\mathbf{R}}$  in (A.7) has structurally the same form as the covariance matrix for some noncoherent set of  $K + J$  signals. Hence, the eigenstructure-based techniques can be applied to this smoothed matrix irrespective of the coherence of the original set of signals to successfully estimate their directions of arrival. This completes the proof.

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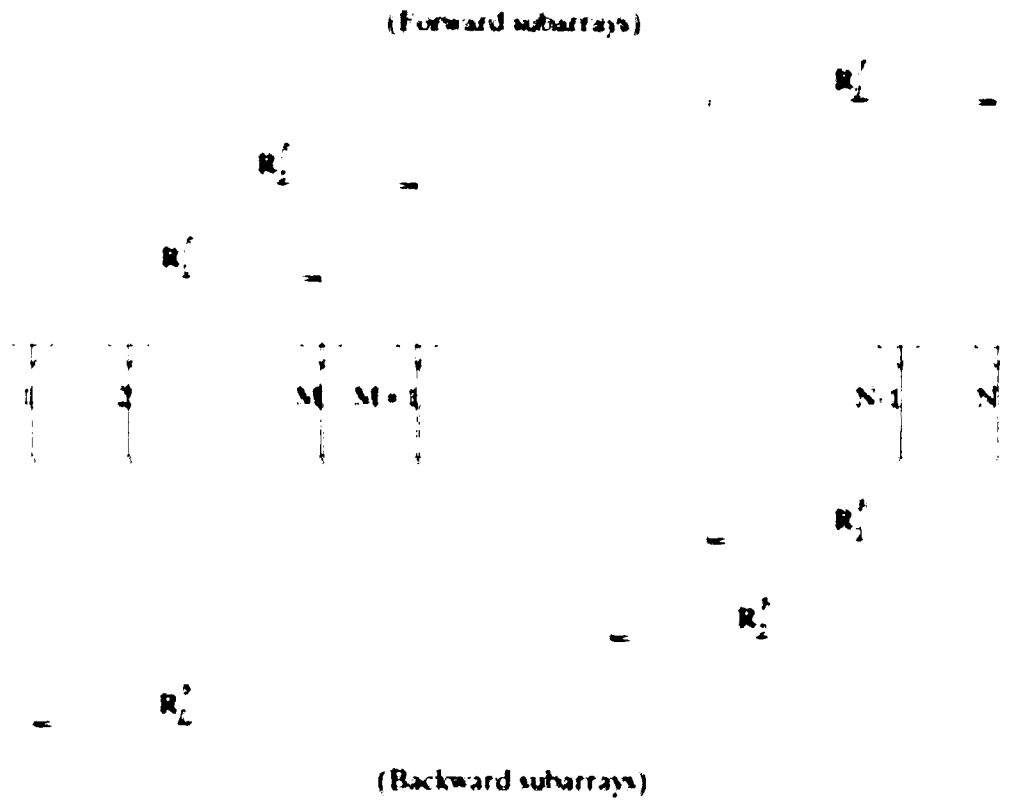


Fig. 1 The Forward/Backward Spatial Smoothing Scheme

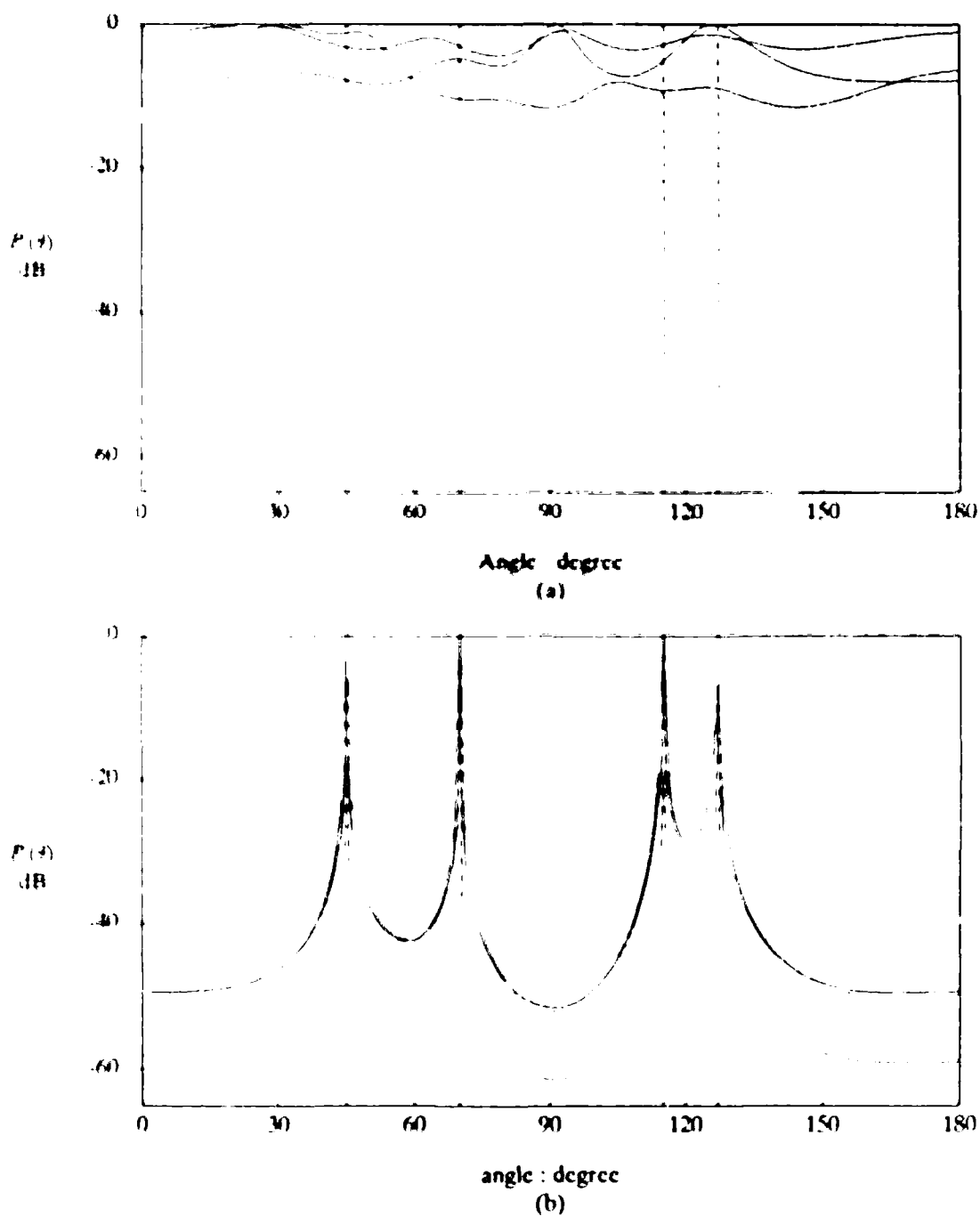


Fig. 2 Direction finding in a coherent scene. A six-element uniform array receives signals from four coherent sources with multipath coefficients  $(0.4, 0.8)$ ,  $(1, 0)$ ,  $(-0.3, -0.7)$  and  $(0.5, -0.6)$ . The arrival angles of the four coherent signals are  $45^\circ$ ,  $70^\circ$ ,  $115^\circ$  and  $127^\circ$ . Input SNR of the reference signal is 5 dB. Three hundred data samples are used to estimate the covariance matrix. (a)  $P(\theta)$  using the conventional MUSIC scheme. (b)  $P(\theta)$  using the forward/backward smoothing scheme.

## 2.2 Performance Analysis of MUSIC-Type High Resolution Estimators for Direction Finding in Correlated and Coherent Scenes

### 2.2.1 Introduction

Eigenstructure-based techniques that yield high resolution have been a topic of great interest in array signal processing since the well-known works of Pisarenko [1], Schmidt [2] and others [3]–[8]. These methods, in general, utilize certain eigenstructure properties resulting from the special structure of the sensor array output covariance matrix for planar wavefronts [2] to generate spectral peaks (or equivalently spectral nulls) along the actual directions of arrival and are known to yield high resolution even when some of the sources are partially correlated. In coherent situations, such as multipath propagation, a direct application of these techniques results in ambiguity, and specific modifications have been suggested to remedy this problem [9]–[11]. Of these, the spatial-smoothing scheme, originally suggested by Evans *et al.* [9] and studied by Shan *et al.* [10], is based on a preprocessing scheme that for a uniform array, partitions the total array of sensors into subarrays and then generates the average of the subarray output covariance matrices. Further, in the case of independent and identical sensor noise, this matrix is shown to be structurally equivalent to that in some correlated scene thereby making it amiable to the above mentioned methods. However, this smoothing scheme - we call it the forward-only smoothing scheme - requires  $2K$  sensor elements to estimate  $K$  coherent directions of arrival. In contrast to this, a modified scheme also proposed by Evans *et al.* - we call it the forward/backward smoothing scheme - uses both forward and backward subarrays for averaging and is shown to require only  $[3K/2]^{(1)}$  sensors to estimate any  $K$  directions of arrival [9, 12].

When the exact ensemble array output covariances are used, all these methods result in unbiased values (i.e., zero for the null spectrum) along the true arrival angles irrespective of signal-to-noise ratios (SNRs) and angular separations of sources.

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(1) The symbol  $[x]$  stands for the integer part of  $x$ .



However, when these covariances are estimated from a finite number of independent snapshots, these techniques exhibit deviations from their ensemble average values. These deviations depend on the specific scheme under consideration together with the SNRs and other signal and array specifications. All this taken together determine the resolution capacity of the technique under consideration.

This report presents a performance analysis of these smoothing techniques when covariances estimated from a large sample size are used in place of their ensemble averages. The analysis for the general forward/backward scheme is carried out first while the forward-only scheme and the conventional unsmoothed (MUSIC) scheme are derived as special cases.

This report is organized as follows: For the sake of completeness, the forward-only and the forward/backward smoothing schemes are summarized in section 2.2.2. Using results derived in Appendix A, section 2.2.3 presents the first-order approximations to the mean and variance of the null spectrum corresponding to the forward/backward, the forward-only and the standard MUSIC schemes. The bias expression in the case of the forward/backward scheme is used to obtain a resolution threshold for two completely coherent, equipowered plane wave sources in white noise; and this result is compared to the resolution threshold obtained by Kaveh *et al.* [13] for two uncorrelated, equipowered plane wave sources in white noise. Finally, in Appendix B, several identities that are found to be useful for performance analysis are developed.

### 2.2.2 Problem Formulation

Let a uniform array consisting of  $M$  sensors receive signals from  $K$  narrowband sources  $u_1(t), u_2(t), \dots, u_{K_o}(t), u_{K_o+1}, \dots, u_K(t)$ , of which the first  $K_o$  signals are completely coherent and the last  $(K - K_o + 1)$  signals are partially correlated. Thus the coherent signals are partially correlated with the remaining set of signals. Further, the respective arrival angles are assumed to be  $\theta_1, \theta_2, \dots, \theta_{K_o}, \theta_{K_o+1}, \dots, \theta_K$  with respect to the line of the array. At any instant, the first  $K_o$  signals  $u_1(t), u_2(t), \dots, u_{K_o}(t)$ ,

are phase-delayed, amplitude-weighted replicas of one of them - say, the first - and hence

$$u_k(t) = \alpha_k u_1(t), \quad k = 1, 2, \dots, K_o \quad (1)$$

where  $\alpha_k$  represents the complex attenuation of the  $k^{th}$  signal with respect to the first signal,  $u_1(t)$ . Using complex signal representation, the received signal  $x_i(t)$  at the  $i^{th}$  sensor can be expressed as

$$x_i(t) = \sum_{k=1}^K u_k(t) \exp(-j\pi(i-1)\cos\theta_k) + n_i(t). \quad (2)$$

Here the interelement distance is taken to be half the wavelength common to all signals and  $n_i(t)$  represents the additive noise at the  $i^{th}$  sensor. It is assumed that the signals and noises are stationary, zero mean circular Gaussian<sup>(2)</sup> independent random processes, and further, the noises are assumed to be independent and identical between themselves with common variance  $\sigma^2$ . Rewriting (2) in common vector notation and with  $\omega_k = \pi \cos \theta_k$  ;  $k = 1, 2, \dots, K$ , we have

$$\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_M(t)]^T = \mathbf{A} \mathbf{u}(t) + \mathbf{n}(t) \quad (3)$$

where

$$\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_K(t)]^T, \quad \mathbf{n}(t) = [n_1(t), n_2(t), \dots, n_M(t)]^T \quad (4)$$

and

$$\mathbf{A} = \sqrt{M} [\mathbf{a}(\omega_1), \mathbf{a}(\omega_2), \dots, \mathbf{a}(\omega_K)] \quad (5)$$

Here  $\mathbf{a}(\omega_k)$  is the normalized direction vector associated with the arrival angle  $\theta_k$ ; i.e.,

(2) A complex random vector  $\mathbf{z}$  is defined to be circular Gaussian if its real part  $\mathbf{x}$  and imaginary part  $\mathbf{y}$  are jointly Gaussian and their joint covariance matrix has the form [14,17]

$$E \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \begin{bmatrix} \mathbf{x}^T & \mathbf{y}^T \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{V} & -\mathbf{W} \\ \mathbf{W} & \mathbf{V} \end{bmatrix}$$

where  $\mathbf{z} = \mathbf{x} + j\mathbf{y}$ . When  $\mathbf{z}$  has zero mean, its covariance matrix is given by  $E[\mathbf{z}\mathbf{z}^\dagger] \triangleq E[(\mathbf{x} + j\mathbf{y})(\mathbf{x}^T - j\mathbf{y}^T)] = \mathbf{V} + j\mathbf{W}$ . Clearly,  $E[\mathbf{z}\mathbf{z}^T] = \mathbf{O}$ . Here onwards T and  $\dagger$  represent the transpose and complex conjugate transpose, respectively.

$$\mathbf{a}(\omega_k) = \frac{1}{\sqrt{M}} \left[ 1, \exp(-j\omega_k), \exp(-j2\omega_k), \dots, \exp(-j(M-1)\omega_k) \right]^T. \quad (6)$$

Notice that  $\mathbf{A}$  is an  $M \times K$  Vandermonde-type matrix ( $M > K$ ) of rank  $K$  and from our assumptions it follows that the array output covariance matrix has the form

$$\mathbf{R} = E [\mathbf{x}(t) \mathbf{x}^\dagger(t)] = \mathbf{A} \mathbf{R}_u \mathbf{A}^\dagger + \sigma^2 \mathbf{I}. \quad (7)$$

The source covariance matrix  $\mathbf{R}_u = E [\mathbf{u}(t) \mathbf{u}^\dagger(t)]$  remains as nonsingular so long as the sources are at most partially correlated. In that case  $\mathbf{A} \mathbf{R}_u \mathbf{A}^\dagger$  is also of rank  $K$  and hence, if  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M$  and  $\beta_1, \beta_2, \dots, \beta_M$  are the eigenvalues and the corresponding eigenvectors of  $\mathbf{R}$  respectively, i.e.,

$$\mathbf{R} = \sum_{l=1}^M \lambda_l \beta_l \beta_l^\dagger, \quad (8)$$

then the above rank property implies that  $\lambda_{K+1} = \lambda_{K+2} = \dots = \lambda_M = \sigma^2$  and consequently  $\beta_i^\dagger \mathbf{a}(\omega_k) = 0$ ,  $i = K+1, K+2, \dots, M$ ;  $k = 1, 2, \dots, K$ . Hence the nulls of  $Q(\omega)$  given by

$$Q(\omega) = \sum_{k=K+1}^M |\beta_k^\dagger \mathbf{a}(\omega)|^2 = 1 - \sum_{k=1}^K |\beta_k^\dagger \mathbf{a}(\omega)|^2 \quad (9)$$

correspond to the actual directions of arrival. However, when some signals are coherent as in (1),  $\mathbf{R}_u$  is singular and the above conclusions are no longer true.

To circumvent this crucial missing rank problem, in the forward-only spatial smoothing scheme[9, 10] a set of overlapping forward subarrays are generated by dividing a large uniform array with  $M_o$  sensors into sets of size  $M$ . Let  $\mathbf{x}_l^f(t)$  stand for the output of the  $l^{\text{th}}$  subarray for  $l = 1, 2, \dots, L \triangleq M_o - M + 1$  where  $L$  denotes the total number of these forward subarrays. Then the covariance matrix of the  $l^{\text{th}}$  subarray is given by

$$\mathbf{R}_l^f = E [\mathbf{x}_l^f(t) (\mathbf{x}_l^f(t))^\dagger] = \mathbf{A} \mathbf{R}_{u,l}^f \mathbf{A}^\dagger + \sigma^2 \mathbf{I} \quad (10)$$

where  $\mathbf{R}_{u,l}^f$  is the source covariance matrix (singular) of the  $l^{\text{th}}$  forward subarray and

the average of these subarray covariance matrices has the form

$$\mathbf{R}_f = \frac{1}{L} \sum_{l=1}^L \mathbf{R}_l^f = \mathbf{A} \left( \frac{1}{L} \sum_{l=1}^L \mathbf{R}_{u,l}^f \right) \mathbf{A}^\dagger + \sigma^2 \mathbf{I} \triangleq \mathbf{A} \mathbf{R}_u^f \mathbf{A}^\dagger + \sigma^2 \mathbf{I}. \quad (11)$$

Here  $\mathbf{R}_u^f$  represents the smoothed source covariance matrix of rank  $\rho(\mathbf{R}_u^f) = K - K_o + \min(K_o, L)$ . Thus if  $L = M_o - M + 1 \geq K_o$  the smoothed source covariance matrix is nonsingular and  $\mathbf{R}_f$  has exactly the same form as the covariance matrix in some noncoherent situation.

To improve upon the number of extra elements needed for smoothing, a set of  $L$  additional backward subarrays are generated in [9, 12] from the same set of sensors by grouping elements at  $M_o, M_o - 1, \dots, M_o - M + 1$  to form the first backward subarray, etc. Let  $\mathbf{x}_l^b(t)$  denote the complex conjugate of the output of the  $l^{\text{th}}$  backward subarray and  $\mathbf{R}_l^b$  the corresponding subarray covariance matrix for  $l = 1, 2, \dots, L$ . Then

$$\mathbf{R}_l^b = E \left[ \mathbf{x}_l^b(t) (\mathbf{x}_l^b(t))^\dagger \right] = \mathbf{A} \mathbf{R}_{u,l}^b \mathbf{A}^\dagger + \sigma^2 \mathbf{I} \quad (12)$$

where  $\mathbf{R}_{u,l}^b$  is the source covariance matrix of the  $l^{\text{th}}$  backward subarray and the average of the subarray covariance matrices has the form

$$\mathbf{R}_b = \frac{1}{L} \sum_{l=1}^L \mathbf{R}_l^b = \mathbf{A} \left( \frac{1}{L} \sum_{l=1}^L \mathbf{R}_{u,l}^b \right) \mathbf{A}^\dagger + \sigma^2 \mathbf{I} \triangleq \mathbf{A} \mathbf{R}_u^b \mathbf{A}^\dagger + \sigma^2 \mathbf{I}. \quad (13)$$

Combining the forward and backward smoothing schemes together, we define the forward/backward (f/b) smoothed covariance matrix as

$$\tilde{\mathbf{R}} \triangleq \frac{1}{2} (\mathbf{R}_f + \mathbf{R}_b) = \mathbf{A} \tilde{\mathbf{R}}_u \mathbf{A}^\dagger + \sigma^2 \mathbf{I} = \sum_{l=1}^M \tilde{\lambda}_l \tilde{\beta}_l \tilde{\beta}_l^\dagger \quad (14)$$

where the smoothed source covariance matrix

$$\tilde{\mathbf{R}}_u = \frac{1}{2} (\mathbf{R}_u^f + \mathbf{R}_u^b) \quad (15)$$

can be shown [12] to have rank  $\rho(\tilde{\mathbf{R}}_u) = K - K_o + \min(K_o, 2L)$ . So, for  $\tilde{\mathbf{R}}_u$  to possess

full rank,  $2L \geq K_o$  or  $2M_o \geq 2M + K_o - 2$ . Recalling that in a coherent situation ( $K = K_o$ ),  $M$  must be at least  $K + 1$ , it follows that the minimum number of sensors must be at least  $\lceil 3K/2 \rceil$ . More precisely, the above conclusion is valid, provided that whenever equality holds among some of the member of the set  $\{\epsilon_k\}_{k=1}^K$  with  $\epsilon_k = (\alpha_k^*/\alpha_k) \exp(j(M_o - 1)\omega_k)$ , the largest subset with equal entries is at most of size  $L$  [12]. Under these conditions  $\tilde{\mathbf{R}}_u$  is of rank  $K$  and hence the eigenvalues of  $\tilde{\mathbf{R}}$  satisfy  $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \dots \geq \tilde{\lambda}_K > \tilde{\lambda}_{K+1} = \tilde{\lambda}_{K+2} = \dots = \tilde{\lambda}_M = \sigma^2$ . Consequently, as in (9) the nulls of  $\tilde{\mathbf{Q}}(\omega)$  given by

$$\tilde{\mathbf{Q}}(\omega) = \sum_{k=K+1}^M |\tilde{\beta}_k^\dagger \mathbf{a}(\omega)|^2 = 1 - \sum_{k=1}^K |\tilde{\beta}_k^\dagger \mathbf{a}(\omega)|^2 \quad (16)$$

correspond to the actual directions of arrival<sup>(3)</sup>.

So far we have assumed that an ensemble average of the array output covariances are available. Generally, a finite data sample is used and estimation is carried out for the unknowns of interest using the maximum likelihood procedure.

For zero mean  $M$ -variate (circular) Gaussian data  $\mathbf{x}(t_n)$ ;  $n = 1, 2, \dots, N$  in (3), with unknown  $M \times M$  covariance matrix  $\mathbf{R}$ , the maximum likelihood (ML) estimate  $\mathbf{S}$  of the covariance matrix is given by [16, 17]

$$\mathbf{S} = \frac{1}{N} \sum_{n=1}^N \mathbf{x}(t_n) \mathbf{x}^\dagger(t_n). \quad (17)$$

Using the invariant property of the maximum likelihood procedure, the corresponding estimates  $\mathbf{S}_f$ ,  $\mathbf{S}_b$  and  $\tilde{\mathbf{S}}$  for the unknown smoothed matrices  $\mathbf{R}_f$ ,  $\mathbf{R}_b$  and  $\tilde{\mathbf{R}}$  can be constructed from  $\mathbf{S}$  by the same rule that is used in constructing  $\mathbf{R}_f$ ,  $\mathbf{R}_b$  and  $\tilde{\mathbf{R}}$ , respectively, from  $\mathbf{R}$ . Thus, for example,

$$\tilde{\mathbf{S}} = \frac{1}{2} (\mathbf{S}_f + \mathbf{S}_b) \quad (18)$$

and

(3) Notice that these arguments can be readily extended to several groups of signals where the signals are coherent within each group, but incoherent between groups. In that case the additional number of sensors required to estimate all incoming arrival angles can be shown to be  $\lceil K_{\max}/2 \rceil$ , where  $K_{\max}$  represents the size of the largest coherent group of signals.

$$\mathbf{S}_f = \frac{1}{L} \sum_{l=1}^L \mathbf{S}_l^f = \frac{1}{NL} \sum_{l=1}^L \sum_{n=1}^N \mathbf{x}_l^f(n) (\mathbf{x}_l^f(n))^\dagger. \quad (19)$$

In what follows we study the statistical properties of these estimated smoothed covariance matrices and their associated sample estimators for direction finding.

### 2.2.3 Performance Analysis

#### 2.2.3.A. Main Results

In this section the statistical behavior of the forward-only and forward/backward smoothing schemes is examined. These results are made use of in deriving expressions for the bias and the resolution threshold of two equipowered coherent sources, and comparisons are made with similar results obtained for uncorrelated sources [13]. Towards this purpose, consider the eigen-representation

$$\tilde{\mathbf{S}} = \tilde{\mathbf{E}} \tilde{\mathbf{L}} \tilde{\mathbf{E}}^\dagger \quad (20)$$

for the ML estimate of the f/b smoothed matrix  $\tilde{\mathbf{R}}$ , with

$$\tilde{\mathbf{E}} = [\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \dots, \tilde{\mathbf{e}}_K, \tilde{\mathbf{e}}_{K+1}, \dots, \tilde{\mathbf{e}}_M]$$

$$\tilde{\mathbf{L}} = \text{diag} [\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_K, \tilde{l}_{K+1}, \dots, \tilde{l}_M]$$

and

$$\tilde{\mathbf{E}} \tilde{\mathbf{E}}^\dagger = \mathbf{I}_M$$

where  $\tilde{e}_{ii} \geq 0$ ,  $i = 1, 2, \dots, M$  for uniqueness. Here the normalized vectors  $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \dots, \tilde{\mathbf{e}}_K$  are the ML estimates of the eigenvectors  $\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_K$  of  $\tilde{\mathbf{R}}$  respectively. Similarly,  $\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_K$  are the ML estimates of the  $K$  largest and distinct eigenvalues  $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_K$  and the mean of  $\tilde{l}_{K+1}, \dots, \tilde{l}_M$  is the sample estimate of the repeating lowest eigenvalue  $\sigma^2$  of  $\tilde{\mathbf{R}}$ . Following (16), the sample direction estimator can be written as

$$\hat{Q}(\omega) = \sum_{k=K+1}^M |\tilde{\mathbf{e}}_k^\dagger \mathbf{a}(\omega)|^2 = 1 - \sum_{k=1}^K |\tilde{\mathbf{e}}_k^\dagger \mathbf{a}(\omega)|^2. \quad (21)$$

The asymptotic distribution of the estimates of the eigenvalues and eigenvectors

associated with the distinct eigenvalues of  $\tilde{\mathbf{R}}$  is derived in (A.31) - (A.32) Appendix A. Corresponding results for the forward-only scheme in (11) and the conventional scheme in (8) can be readily evaluated as special cases of this general result. It is also shown there that the estimated eigenvalues and a specific set of corresponding unnormalized eigenvectors are asymptotically (in the sense of large  $N$ ) jointly Gaussian with means and covariances as derived there (See (A.33) and (A.34)). Further, after proper renormalization and using an exact relationship among the different sets of eigenvectors, it is shown in Appendix A that (see (A.39))

$$E[\hat{Q}(\omega)] = \bar{Q}(\omega) + \frac{1}{N} \sum_{i=1}^K \left[ \sum_{\substack{k=1 \\ k \neq i}}^M \frac{\bar{\Gamma}_{iikk}}{(\bar{\lambda}_i - \bar{\lambda}_k)^2} |\tilde{\beta}_i^\dagger \mathbf{a}(\omega)|^2 - \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq i}}^M \frac{\bar{\Gamma}_{iikl}}{(\bar{\lambda}_i - \bar{\lambda}_k)(\bar{\lambda}_i - \bar{\lambda}_l)} \mathbf{a}^\dagger(\omega) \tilde{\beta}_k \tilde{\beta}_l^\dagger \mathbf{a}(\omega) \right] + o(1/N^2) \quad (22)$$

where from (A.16)

$$\begin{aligned} \bar{\Gamma}_{iklj} \triangleq & \frac{1}{4L^2} \sum_{p=1}^L \sum_{q=1}^L \left[ \tilde{\beta}_i^\dagger \mathbf{R}_{pq}^f \tilde{\beta}_k \tilde{\beta}_l^\dagger \mathbf{R}_{qp}^f \tilde{\beta}_j + \tilde{\beta}_i^\dagger \mathbf{R}_{pq}^b \tilde{\beta}_k \tilde{\beta}_l^\dagger \mathbf{R}_{qp}^b \tilde{\beta}_j \right. \\ & \left. + \tilde{\beta}_i^\dagger \mathbf{R}_{pq_0}^f \tilde{\gamma}_l \tilde{\gamma}_j^\dagger \mathbf{R}_{p_0q}^b \tilde{\beta}_k + \tilde{\beta}_i^\dagger \mathbf{R}_{pq_0}^b \tilde{\gamma}_l \tilde{\gamma}_j^\dagger \mathbf{R}_{p_0q}^f \tilde{\beta}_k \right] \end{aligned} \quad (23)$$

with  $\tilde{\gamma}_i, i = 1, 2, \dots, M$  as defined there and  $\mathbf{R}_{pq}^f, \mathbf{R}_{pq}^b$  as in (A.17).

Similar bias expressions for the forward-only smoothing scheme can be obtained from (22) by replacing (23) with (A.18). In particular for the conventional (unsmoothed) MUSIC case with (A.19) in (23), after some simplifications (22) reduces to

$$\begin{aligned} E[\hat{Q}(\omega)] &= Q(\omega) + \frac{1}{N} \sum_{i=1}^K \left[ \sum_{\substack{k=1 \\ k \neq i}}^M \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} \left( |\beta_i^\dagger \mathbf{a}(\omega)|^2 - |\beta_k^\dagger \mathbf{a}(\omega)|^2 \right) \right] + o(1/N^2) \\ &= Q(\omega) + \frac{1}{N} \sum_{i=1}^K \frac{\lambda_i \sigma^2}{(\lambda_i - \sigma^2)^2} \left( (M - K) |\beta_i^\dagger \mathbf{a}(\omega)|^2 - Q(\omega) \right) + o(1/N^2) \end{aligned} \quad (24)$$

where  $\lambda_i, \beta_i, i = 1, 2, \dots, M$ , are as defined in (8) and  $Q(\omega)$  is given by (9).

Similarly, from (A.42)

$$\begin{aligned} \text{Var}(\hat{Q}(\omega)) \triangleq \hat{\sigma}^2(\omega) = \\ \frac{2}{N} \sum_{i=1}^K \sum_{j=1}^K \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq j}}^M \frac{\text{Re} \left[ \left( \bar{\Gamma}_{klji} \mathbf{a}^\dagger(\omega) \bar{\beta}_j \bar{\beta}_i^\dagger \mathbf{a}(\omega) + \bar{\Gamma}_{kjli} \mathbf{a}^\dagger(\omega) \bar{\beta}_l \bar{\beta}_j^\dagger \mathbf{a}(\omega) \right) \mathbf{a}^\dagger(\omega) \bar{\beta}_k \bar{\beta}_i^\dagger \mathbf{a}(\omega) \right]}{(\bar{\lambda}_i - \bar{\lambda}_k)(\bar{\lambda}_j - \bar{\lambda}_l)} \\ + o(1/N^2) \end{aligned} \quad (25)$$

which for the conventional MUSIC case reduces to

$$\begin{aligned} \text{Var}(\hat{Q}(\omega)) \triangleq \sigma^2(\omega) &= \frac{2}{N} \sum_{i=1}^K \left[ \sum_{\substack{k=1 \\ k \neq i}}^M \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} |\beta_i^\dagger \mathbf{a}(\omega)|^2 |\beta_k^\dagger \mathbf{a}(\omega)|^2 \right. \\ &\quad \left. - \sum_{\substack{j=1 \\ j \neq i}}^K \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} |\beta_i^\dagger \mathbf{a}(\omega)|^2 |\beta_j^\dagger \mathbf{a}(\omega)|^2 \right] + o(1/N^2) \\ &= \frac{2}{N} \sum_{i=1}^K \sum_{k=K+1}^M \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} |\beta_i^\dagger \mathbf{a}(\omega)|^2 |\beta_k^\dagger \mathbf{a}(\omega)|^2 + o(1/N^2) \\ &= \frac{2}{N} Q(\omega) \sum_{i=1}^K \frac{\lambda_i \sigma^2}{(\lambda_i - \sigma^2)^2} |\beta_i^\dagger \mathbf{a}(\omega)|^2 + o(1/N^2). \end{aligned} \quad (26)$$

Since along the actual arrival angles,  $Q(\omega_k) = 0$ ,  $k = 1, 2, \dots, K$ , (26) allows us to conclude that within the above approximation,

$$\sigma^2(\omega_k) = 0 \quad ; \quad k = 1, 2, \dots, K \quad (27)$$

i.e., in all multiple target situations, where the conventional MUSIC scheme is applicable, the variance of the estimator in (21) along the true arrival angles is zero within a first-order approximation. Although at first this conclusion does not seem to agree



with the results of Kaveh *et al.*, for a two-source case (See (30) in [13]), an algebraic manipulation shows that within the above approximation, their  $Var(\hat{D}(\omega_k)) = 0$ , agreeing with this result.

The general expressions for bias and variance in (22) and (25) can be used to determine the required sample size for a certain performance level or to arrive at useful resolution criteria for the forward-only or the f/b smoothing schemes. Though the general cases are often intractable, a complete analysis is possible for the f/b scheme with  $L = 1$ , which of course can decorrelate and resolve two coherent sources. As shown in the next section, this case leads to some interesting results, including the resolution threshold for two completely coherent equipowered sources in white noise.

### 2.2.3.B Two-Source Case

Consider the special case, where the two sources present in the scene are completely coherent with each other. In that situation, the array output data together with its complex conjugated backward version  $\mathbf{x}_i^b(n)$ ;  $n = 1, 2, \dots, N$ , (f/b smoothing with  $L = 1$ ) can be used to decorrelate the incoming signals, thereby making it possible to estimate their arrival angles. For two equipowered sources, the bias and variance of the associated sample estimator can be computed by using (A.20) in (22) and (25), respectively. After a series of algebraic manipulations for mean value of the estimator we have

$$E[\hat{Q}(\omega)] = \bar{Q}(\omega) + \frac{1}{2N} \sum_{i=1}^2 \frac{\bar{\lambda}_i \sigma^2}{(\bar{\lambda}_i - \sigma^2)^2} \left[ (M - 2) |\hat{\beta}_i^\dagger \mathbf{a}(\omega)|^2 - \bar{Q}(\omega) \right] + o\left(\frac{1}{N^2}\right). \quad (28)$$

Let  $\bar{\eta}(\omega)$  and  $\eta(\omega)$  denote the bias in the f/b smoothing scheme and the conventional MUSIC scheme. Then from (28)

$$\begin{aligned} \bar{\eta}(\omega) &\triangleq E[\hat{Q}(\omega)] - \bar{Q}(\omega) \\ &= \frac{1}{2N} \sum_{i=1}^2 \frac{\bar{\lambda}_i \sigma^2}{(\bar{\lambda}_i - \sigma^2)^2} \left[ (M - 2) |\hat{\beta}_i^\dagger \mathbf{a}(\omega)|^2 - \bar{Q}(\omega) \right] + o(1/N^2) \end{aligned} \quad (29)$$

and from (24) with  $K = 2$  we have

$$\eta(\omega) = \frac{1}{N} \sum_{i=1}^2 \frac{\lambda_i \sigma^2}{(\lambda_i - \sigma^2)^2} \left[ (M-2) |\beta_i^\dagger \mathbf{a}(\omega)|^2 - Q(\omega) \right] + o(1/N^2). \quad (30)$$

To evaluate variance, let  $q_{ijkl}$  denote a typical term in (25). Then

$$\bar{\sigma}^2(\omega) = \frac{2}{N} (q_{1122} + q_{1221} + q_{2112} + q_{2211}) + \frac{2}{N} \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=3}^M \sum_{l=3}^M q_{ijkl}.$$

Using (A.20) it is easy to show that the terms within the first parentheses add up to zero, and by repeated use of (A.20) over the remaining terms it can be shown that

$$\begin{aligned} \bar{\sigma}^2(\omega) &= \frac{1}{N} \sum_{i=1}^2 \frac{\bar{\lambda}_i \sigma^2}{(\bar{\lambda}_i - \sigma^2)^2} \left( |\bar{\beta}_i^\dagger \mathbf{a}(\omega)|^2 \bar{Q}(\omega) \right. \\ &\quad \left. + \sum_{k=3}^M \operatorname{Re} \left[ \bar{\beta}_k^\dagger \mathbf{a}(\omega) \bar{\gamma}_k^\dagger \mathbf{a}(\omega) \mathbf{a}^\dagger(\omega) \bar{\beta}_i \mathbf{a}^\dagger(\omega) \bar{\gamma}_i \right] \right) + o(1/N^2) \\ &= \frac{2 \bar{Q}(\omega)}{N} \sum_{i=1}^2 \frac{\bar{\lambda}_i \sigma^2}{(\bar{\lambda}_i - \sigma^2)^2} |\bar{\beta}_i^\dagger \mathbf{a}(\omega)|^2 + o(1/N^2). \end{aligned} \quad (31)$$

Again  $\bar{\sigma}^2(\omega_1) = \bar{\sigma}^2(\omega_2) = 0$  in this case also. Notice that bias in (29) and (30) for the f/b smoothing scheme with  $L = 1$  and the conventional MUSIC scheme with  $K = 2$  are functionally identical except for a multiplication factor of two in the conventional case. Moreover, these results suggest that in a correlated two source case, the f/b scheme will perform superior to the conventional one. This can be easily illustrated in an uncorrelated scene where equality of the array output covariance matrices in the smoothed and conventional cases implies  $\lambda_i = \bar{\lambda}_i$ ,  $\beta_i = \bar{\beta}_i$ ,  $i = 1, 2, \dots, M$  and  $\bar{\Gamma}_{uu} = (\lambda_1 \lambda_2 / 2) (\delta_{ik} \delta_{ij} + \beta_i^\dagger \gamma_i \gamma_j^\dagger \beta_k)$ . Substituting these values in (22) and (25), it easily follows that  $\bar{\eta}(\omega) = \eta(\omega)/2$  and  $\bar{\sigma}^2(\omega) = \sigma^2(\omega)$  and consequently, the f/b scheme is uniformly superior to the conventional one. However, in a correlated scene, although the effective correlation coefficient reduces in magnitude after smoothing, it is difficult to

exhibit such uniform superior behavior explicitly. Nevertheless, certain clarifications are possible. To see this, consider two cases, the first one consisting of two correlated sources with correlation coefficient  $\rho_o$  and the second one consisting of two coherent sources. In both cases, the sources are of equal power and have the same arrival angles  $\theta_1$  and  $\theta_2$ . The correlated case can be resolved using the conventional MUSIC scheme, and the coherent scene can be decorrelated and resolved using the f/b smoothing scheme with  $L = 1$ . In the latter case from (B.2) and in particular from (B.9), smoothing results in an effective correlation coefficient  $\rho_i = \exp(j(M - 1)\omega_d) \cos(M - 1)\omega_d$  with  $\omega_d = \pi (\cos\theta_1 - \cos\theta_2)/2$ , between the sources. In the event when the temporal correlation  $\rho_o$  in the conventional case is equal to the above  $\rho_i$ , then  $\mathbf{R} = \bar{\mathbf{R}}$ ,  $\lambda_i = \bar{\lambda}_i$ ;  $\beta_i = \bar{\beta}_i$ ,  $i = 1, 2$ ; and from (26), (29) - (31) the f/b scheme is uniformly superior to the conventional one in terms of bias. This conclusion is also supported by simulation results presented in Fig. 1 with details as indicated there.

As one would expect, for closely spaced sources the performance of the conventional scheme in an uncorrelated source scene is superior to that of the f/b scheme in a coherent scene. This is to be expected because for small values of angular separation ( $\Delta^2 < 1$ ) from (B.28) and (B.29), we have  $\eta(\omega_i) < \bar{\eta}(\omega_i)$ ,  $i = 1, 2$ . The deviation of  $\eta(\omega_i)$  and  $\bar{\eta}(\omega_i)$  from zero - their nominal value - suggests the loss in resolution for the respective estimators. Within a first-order approximation, since the estimators have zero variance along the two arrival angles in both cases, for a fixed number of samples a threshold in terms of SNR exists below which the two nulls corresponding to the true arrival angles are no longer identifiable. This has led to the definition of the resolution threshold for two closely spaced sources as that value of SNR at which [13]

$$E[\hat{Q}(\omega_1)] = E[\hat{Q}(\omega_2)] = E[\hat{Q}((\omega_1 + \omega_2)/2)], \quad (32)$$

whenever  $Var(\hat{Q}(\omega_1)) \approx Var(\hat{Q}(\omega_2)) \approx 0$ . In the case of the two equipowered uncorrelated sources equating (B.28) and (B.30), Kaveh *et al.* found the resolution threshold to be [13]

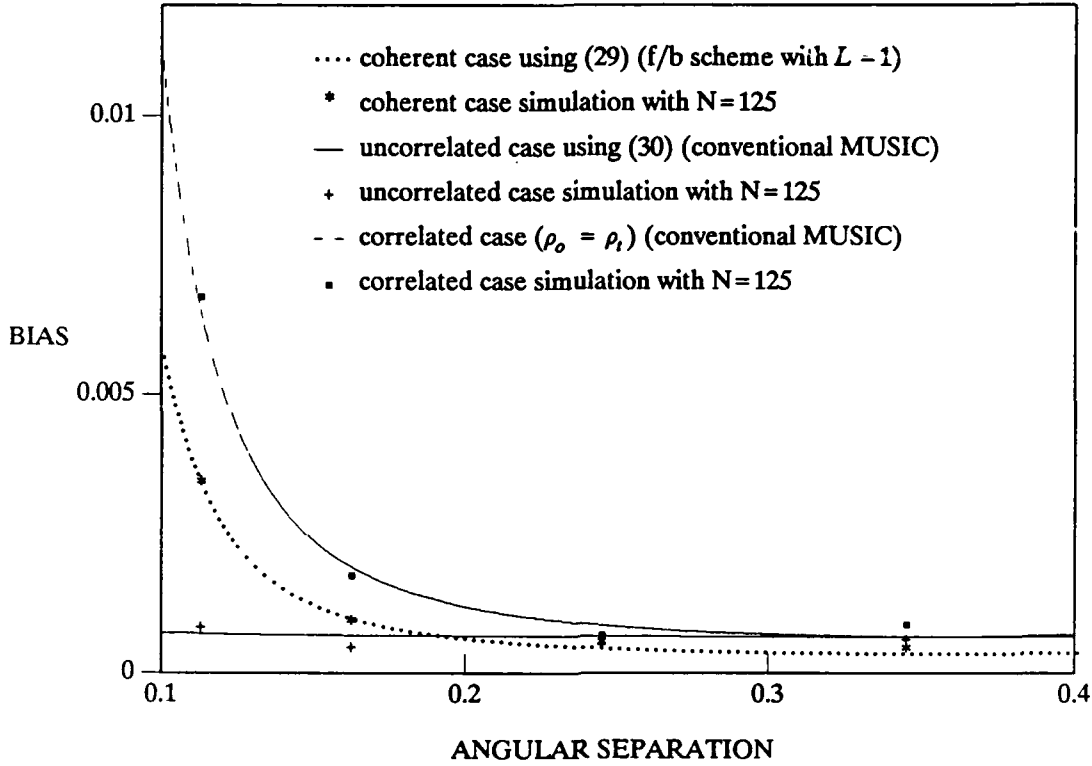


Fig. 1 Bias at one of the arrival angles vs. angular separation for two equipowered sources in uncorrelated, correlated and coherent scenes. A ten element array is used to collect signal in all these cases. Input SNR is taken to be 10dB and number of simulations in each case is 30.

$$\xi_T \approx \frac{1}{N} \left[ \frac{20(M-2)}{\Delta^4} \left( 1 + \left[ 1 + \frac{N}{5(M-2)} \Delta^2 \right]^{1/2} \right) \right]. \quad (33)$$

The corresponding threshold  $\tilde{\xi}_T$  in the coherent case can be found by equating (B.29) and (B.31). In that case, after some algebraic manipulations we have

$$\begin{aligned} \tilde{\xi}_T &\approx \frac{1}{N} \left[ \frac{20(M-2)}{\Delta^4} \left( \frac{1}{3\Delta^2} - \frac{1}{20} - \frac{\Delta^2}{16} \right) \left( 1 + \left[ 1 + \frac{N(M-2)}{5M(M-4)} \Delta^2 \right]^{1/2} \right) \right] \\ &= \left( \frac{1}{3\Delta^2} - \frac{1}{20} - \frac{\Delta^2}{16} \right) \xi_T \approx \left( \frac{1}{M\omega_d} \right)^2 \xi_T. \end{aligned} \quad (34)$$

Though  $\tilde{\xi}_T$  and  $\xi_T$  possess similar features, for small arrays the resolution threshold in the coherent case can be substantially larger than that in the uncorrelated case. This asymptotic analysis is also found to be in agreement with the results obtained by

Monte Carlo simulations. A typical case study is reported in Table 1. When the equality in (32) is true, the probability of resolution was found to range from 0.33 to 0.5 in both cases there. This in turn implies that the above analysis should give an approximate threshold in terms of  $\xi$  for the 0.33 to 0.5 probability of resolution region. Comparisons are carried out in Fig. 2 using (33), (34) and simulation results from Table 1 for 0.33 to 0.5 probability of resolution. Fig. 3 show a similar comparison for yet another array length. In all these cases the close agreement between the theory and simulation results is clearly evident.

The above range (0.33 to 0.5) for the probability of resolution can be explained by reexamining the arguments used in deriving the resolution thresholds (33) and (34). In fact, (32) has been justified by observing that within a first-order approximation,  $Var(\hat{Q}(\omega_1)) = Var(\hat{Q}(\omega_2)) = 0$ . Although  $Var(\hat{Q}((\omega_1 + \omega_2)/2))$  is equally important in that analysis, it is nevertheless nonzero (see B.32 and B.33). This implies that though  $\xi_T$  and  $\bar{\xi}_T$  satisfies (32), in an actual set of trials the estimated mean value of  $\hat{Q}((\omega_1 + \omega_2)/2)$  will almost always be in the interval  $(0, 2E[\hat{Q}((\omega_1 + \omega_2)/2)])$  and clearly resolution of the two nulls in  $\hat{Q}(\omega)$  is possible only if this mean estimate lies in the upper half of the above interval. In the special case of a symmetrical density function for the mean value estimate, this occurs with probability 0.5 and the observed range may be attributed to the skewed nature of the actual probability density function.

#### IV. Conclusions

The asymptotic analysis of a set of high resolution estimators for resolving plane waves that are correlated or coherent with one another is presented here. A forward/backward spatial smoothing scheme that decorrelates coherent arrivals is treated first for its mean and variance; similar expressions for the forward-only smoothing scheme and the unsmoothed conventional MUSIC scheme are derived as special cases of this general analysis. In particular, the variance of the conventional MUSIC estimator along the true arrival angles is shown to be zero within a first-order approximation. This result is independent of the total number of sources present in

Table 1

Resolution threshold and probability of resolution vs. angular separation for two equipowered sources in an uncorrelated scene. (number of sensors = 7, number of snapshots = 100, number of simulations = 100). Probability of Resolution  $\triangleq$  Total number of successes in 100 simulations/100.

angles of arrival		angular separation	uncorrelated		coherent	
$\theta_1$	$\theta_2$	$2\omega_d$	SNR(dB)	Prob.	SNR(dB)	Prob.
19.00	25.00	0.1232	17	0.16	26	0.17
			18	0.17	27	0.39
			19	0.22	28	0.44
			20	0.47	29	0.58
			21	0.51		
			22	0.71		
34.00	40.00	0.1978	11	0.26	15	0.27
			12	0.39	16	0.35
			13	0.60	17	0.61
			14	0.65	18	0.76
113.00	118.00	0.2474	6	0.20	9	0.09
			7	0.30	10	0.36
			8	0.41	11	0.44
			9	0.52	12	0.57
			10	0.74		
60.00	66.00	0.2930	4	0.23	5	0.22
			5	0.37	6	0.36
			6	0.47	7	0.35
			7	0.66	8	0.62
127.00	135.00	0.3308	1	0.18	1	0.11
			2	0.22	2	0.20
			3	0.33	3	0.26
			4	0.53	4	0.47
			5	0.66	5	0.60

the scene. Further, a resolution threshold, which depends on the relative angular separation, number of sensors, number of snapshots and signal-to-noise ratios, for two coherent, equipowered, closely spaced signals is derived from the bias and variance expressions of the f/b smoothing scheme. This large sample based asymptotic result is compared to the one obtained by Kaveh *et al.* [13] for two uncorrelated, equipowered, closely spaced signals. From these comparisons, to detect the arrival angles for two closely spaced, equipowered, coherent signals, under identical conditions approximately  $[(1/M\omega_d)^2 - 1]$  times additional snapshots than those in an uncorrelated situation are required. These results are also seen to closely agree with those obtained from Monte Carlo simulations.

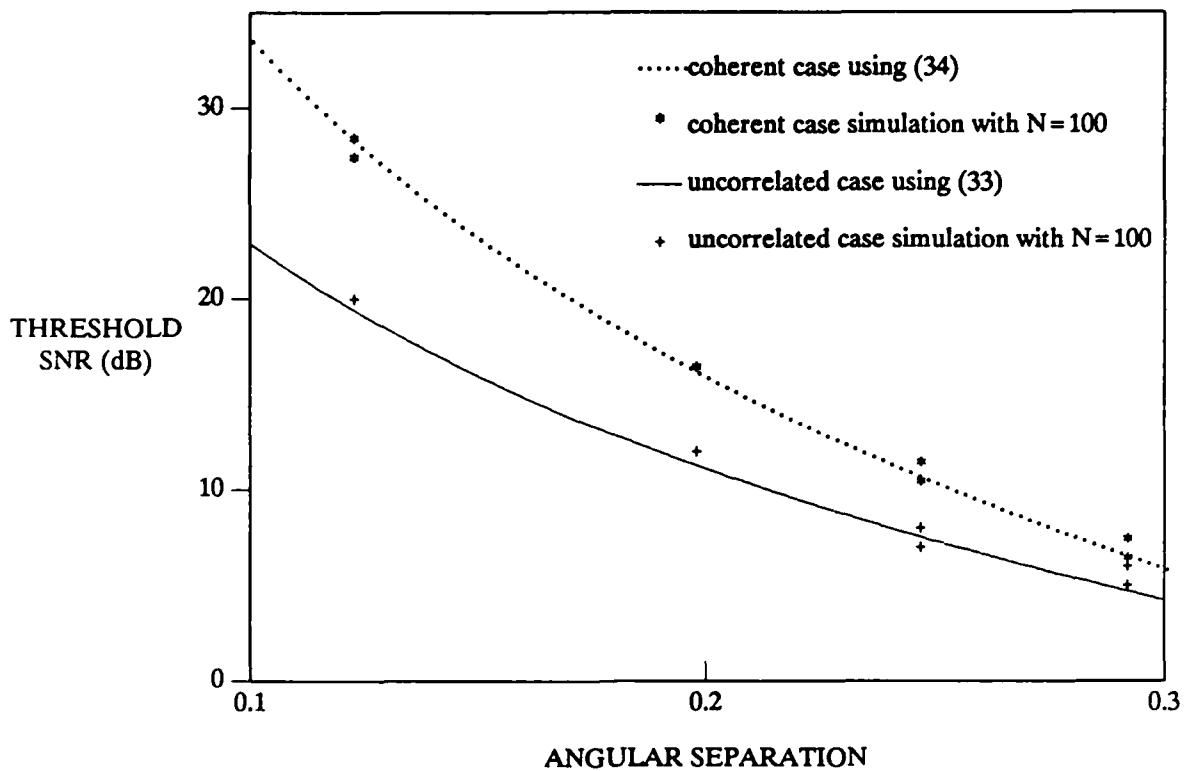


Fig. 2 Resolution threshold vs angular separation for two equipowered sources in coherent and uncorrelated scenes. A seven element array is used to receive signals in both cases.

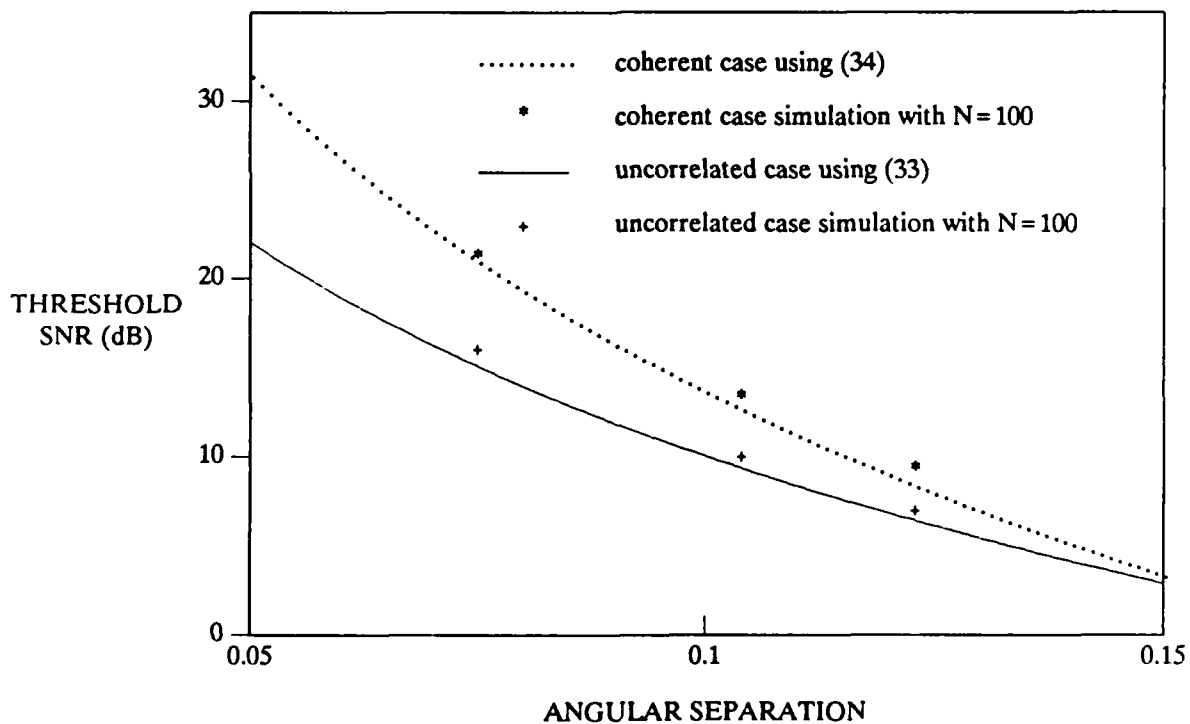


Fig. 3 Resolution threshold vs. angular separation for two equipowered sources in coherent and uncorrelated scenes. A fifteen element array is used to receive signals in both cases.

## Appendix A

### Asymptotic Distribution of the Sample Eigenvalues and Eigenvectors Corresponding to Distinct Eigenvalues of $\tilde{\mathbf{R}}$

With symbols as defined in the text,  $\tilde{\mathbf{S}}$  representing the ML estimate of the forward/backward (f/b) smoothed covariance matrix  $\tilde{\mathbf{R}}$ , we have

$$\tilde{\mathbf{R}} = \frac{1}{2L} \sum_{l=1}^L \left( \mathbf{R}_l^f + \mathbf{R}_l^b \right) \triangleq \tilde{\mathbf{B}} \tilde{\mathbf{A}} \tilde{\mathbf{B}}^\dagger \quad (\text{A.1})$$

$$\tilde{\mathbf{S}} = \frac{1}{2L} \sum_{l=1}^L \left[ \frac{1}{N} \sum_{n=1}^N \left( \mathbf{x}_l^f(n) (\mathbf{x}_l^f(n))^\dagger + \mathbf{x}_l^b(n) (\mathbf{x}_l^b(n))^\dagger \right) \right] = \tilde{\mathbf{E}} \tilde{\mathbf{L}} \tilde{\mathbf{E}}^\dagger \quad (\text{A.2})$$

where

$$\tilde{\mathbf{B}} = [\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_M] \quad , \quad \tilde{\mathbf{E}} = [\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \dots, \tilde{\mathbf{e}}_M]$$

$$\tilde{\mathbf{A}} = \text{diag} [\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_K, \sigma^2, \dots, \sigma^2] \quad , \quad \tilde{\mathbf{L}} = \text{diag} [\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_K, \tilde{l}_{K+1}, \dots, \tilde{l}_M]$$

$$\tilde{\mathbf{B}} \tilde{\mathbf{B}}^\dagger = \tilde{\mathbf{E}} \tilde{\mathbf{E}}^\dagger = \mathbf{I}_M$$

and  $\tilde{\mathbf{B}}, \tilde{\mathbf{E}}$  satisfies  $\tilde{\beta}_{ii}, \tilde{e}_{ii} \geq 0, i = 1, 2, \dots, M$  for uniqueness. As is well known, the eigenvectors are not unique, and let  $\mathbf{C}$  represent yet another set of eigenvectors for  $\tilde{\mathbf{S}}$ , i.e.,

$$\tilde{\mathbf{S}} = \mathbf{C} \tilde{\mathbf{L}} \mathbf{C}^\dagger \quad (\text{A.3})$$

where

$$\mathbf{C} = [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_M] \quad , \quad \mathbf{C} \mathbf{C}^\dagger = \mathbf{I}_M. \quad (\text{A.4})$$

For reasons that will become apparent later,  $\mathbf{C}$  is made unique here by requiring that all diagonal elements of  $\mathbf{Y}$  given by

$$\mathbf{Y} = \tilde{\mathbf{B}}^\dagger \mathbf{C} \quad (\text{A.5})$$

are positive ( $y_{ii} \geq 0, i = 1, 2, \dots, M$ ). In what follows we first derive the asymptotic distribution of the set of sample eigenvectors and eigenvalues of  $\tilde{\mathbf{S}}$  given by (A.3) - (A.5) and use this to analyze the performance of the sample directions of arrival estimator  $\hat{\mathbf{Q}}(\omega)$  in (21). This is made possible by noticing that although the estimated



eigenvectors  $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \dots, \tilde{\mathbf{e}}_K$ , in (21) are structurally similar to their counterparts  $\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_K$ , and in particular have  $\tilde{e}_{ii} \geq 0$ , nevertheless they are also related to  $\mathbf{c}_i, i = 1, 2, \dots, K$ , through a phase factor<sup>(4)</sup>; i.e.,

$$\tilde{\mathbf{e}}_i = e^{j\phi_i} \mathbf{c}_i, \quad i = 1, 2, \dots, K \quad (\text{A.6})$$

and hence

$$\hat{Q}(\omega) = 1 - \sum_{i=1}^K |\tilde{\mathbf{e}}_i^\dagger \mathbf{a}(\omega)|^2 = 1 - \sum_{i=1}^K |\mathbf{c}_i^\dagger \mathbf{a}(\omega)|^2 = 1 - \sum_{i=1}^K y_i(\omega) \quad (\text{A.7})$$

$$y_i(\omega) = |\mathbf{c}_i^\dagger \mathbf{a}(\omega)|^2. \quad (\text{A.8})$$

Thus, the statistical properties of  $\hat{Q}(\omega)$  can be completely specified by those of  $\mathbf{c}_i, i = 1, 2, \dots, K$ , and towards this purpose, let

$$\mathbf{F} = \sqrt{N} (\tilde{\mathbf{L}} - \tilde{\mathbf{A}}) \quad (\text{A.9})$$

$$\mathbf{G} = [\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_K, \dots, \mathbf{g}_M] = \sqrt{N} (\mathbf{C} - \tilde{\mathbf{B}}) \quad (\text{A.10})$$

and

$$\mathbf{T} = \tilde{\mathbf{B}}^\dagger \tilde{\mathbf{S}} \tilde{\mathbf{B}} = \tilde{\mathbf{B}}^\dagger \mathbf{C} \tilde{\mathbf{L}} \mathbf{C}^\dagger \tilde{\mathbf{B}} = \mathbf{Y} \tilde{\mathbf{L}} \mathbf{Y}^\dagger \quad (\text{A.11})$$

where  $\mathbf{Y}$  is as defined in (A.5) with  $y_{ii} \geq 0, i = 1, 2, \dots, M$ . Further, let

$$\begin{aligned} \mathbf{U} &= \sqrt{N} (\mathbf{T} - \tilde{\mathbf{A}}) \\ &= \frac{1}{\sqrt{N}} \sum_{n=1}^N \left[ \frac{1}{2L} \sum_{l=1}^L (\mathbf{z}_l^f(n) (\mathbf{z}_l^f(n))^\dagger + \mathbf{z}_l^b(n) (\mathbf{z}_l^b(n))^\dagger) - \tilde{\mathbf{A}} \right] \end{aligned} \quad (\text{A.12})$$

with

$$\mathbf{z}_l^f(n) = \tilde{\mathbf{B}}^\dagger \mathbf{x}_l^f(n) \sim N(0, \tilde{\mathbf{B}}^\dagger \mathbf{R}_l^f \tilde{\mathbf{B}}); \quad \mathbf{z}_l^b(n) = \tilde{\mathbf{B}}^\dagger \mathbf{x}_l^b(n) \sim N(0, \tilde{\mathbf{B}}^\dagger \mathbf{R}_l^b \tilde{\mathbf{B}}). \quad (\text{A.13})$$

It is easily verified that these random vectors preserve the circular Gaussian property of the data vectors. Again, from the independence of observations, asymptotically every entry in  $\mathbf{U}$  is a sum of a large number of independent random variables. Using

(4) From (A.2) and (A.3) we have  $\tilde{\mathbf{L}} \tilde{\mathbf{V}} = \tilde{\mathbf{V}} \tilde{\mathbf{L}}$  where  $\tilde{\mathbf{V}} = \tilde{\mathbf{E}}^\dagger \mathbf{C}$ . Thus, for any nonrepeating eigenvalue  $\tilde{l}_i$ , it follows that  $\tilde{\mathbf{e}}_i = e^{j\phi_i} \mathbf{c}_i$ . Interestingly, since the eigenvalues of the estimated covariance matrix are always distinct with probability one [18], (A.6) is true for  $1 \leq i \leq M$ .

the multivariate central limit theorem [16], the limiting distribution of  $\mathbf{U}$  tends to be normal with means and covariances given by

$$E[u_{ij}] = \frac{1}{\sqrt{N}} \sum_{n=1}^N E \left[ \frac{1}{2L} \sum_{l=1}^L \left( z_{li}^f(n) z_{lj}^{f*}(n) + z_{li}^b(n) z_{lj}^{b*}(n) \right) - \bar{\lambda}_i \delta_{ij} \right] = 0 \quad (\text{A.14})$$

(Here onwards whenever there is no confusion, we will suppress the time index  $n$ ) since

$$\begin{aligned} E \left[ \frac{1}{2L} \sum_{l=1}^L \left( z_{li}^f (z_{lj}^f)^* + z_{li}^b (z_{lj}^b)^* \right) \right] &= \tilde{\beta}_i^\dagger E \left[ \frac{1}{2L} \sum_{l=1}^L \left( \mathbf{x}_l^f (\mathbf{x}_l^f)^\dagger + \mathbf{x}_l^b (\mathbf{x}_l^b)^\dagger \right) \right] \tilde{\beta}_j \\ &= \tilde{\beta}_i^\dagger E[\tilde{\mathbf{S}}] \tilde{\beta}_j = \tilde{\beta}_i^\dagger \tilde{\mathbf{R}} \tilde{\beta}_j = \bar{\lambda}_i \delta_{ij} \end{aligned} \quad (\text{A.15})$$

and

$$\begin{aligned} E[u_{ij} u_{kl}^*] &= \frac{1}{N} \sum_{n=1}^N \left[ \frac{1}{4L^2} \sum_{p=1}^L \sum_{q=1}^L \left\{ E[z_{pi}^f z_{pj}^{f*} z_{qk}^{f*} z_{ql}^f] + E[z_{pi}^f z_{pj}^{f*} z_{qk}^b z_{ql}^{b*}] \right. \right. \\ &\quad \left. \left. + E[z_{pi}^b z_{pj}^{b*} z_{qk}^{f*} z_{ql}^f] + E[z_{pi}^b z_{pj}^{b*} z_{qk}^b z_{ql}^{b*}] \right\} \right] - \bar{\lambda}_i \bar{\lambda}_k \delta_{ij} \delta_{kl}. \end{aligned}$$

Using the results<sup>(5)</sup> for fourth-order moments of jointly circular Gaussian random variables and after some algebraic manipulations, we have

$$\begin{aligned} E[u_{ij} u_{kl}^*] &= \frac{1}{4L^2} \sum_{p=1}^L \sum_{q=1}^L \left( \tilde{\beta}_i^\dagger \mathbf{R}_{pq}^f \tilde{\beta}_k \tilde{\beta}_l^\dagger \mathbf{R}_{qp}^f \tilde{\beta}_j + \tilde{\beta}_i^\dagger \mathbf{R}_{pq}^b \tilde{\beta}_k \tilde{\beta}_l^\dagger \mathbf{R}_{qp}^b \tilde{\beta}_j \right. \\ &\quad \left. + \tilde{\beta}_i^\dagger \mathbf{R}_{pq_o}^f \tilde{\gamma}_l \tilde{\gamma}_j^\dagger \mathbf{R}_{p_o q}^b \tilde{\beta}_k + \tilde{\beta}_i^\dagger \mathbf{R}_{pq_o}^b \tilde{\gamma}_l \tilde{\gamma}_j^\dagger \mathbf{R}_{p_o q}^f \tilde{\beta}_k \right) \triangleq \bar{\Gamma}_{ijkl}. \end{aligned} \quad (\text{A.16})$$

Here by definition  $p_o = M_o - M - p + 2$ ,  $q_o = M_o - M - q + 2$  and  $\tilde{\gamma}_j$  is the inverted  $\tilde{\beta}_j^*$  vector with  $\tilde{\gamma}_{j,m} = \tilde{\beta}_{j, M-m+1}^*$ . For  $L = 1$ , in an uncorrelated as well as a two equi-powered coherent source scene, it is easy to show that  $\tilde{\beta}_i^\dagger \tilde{\mathbf{R}} \tilde{\beta}_j = \tilde{\gamma}_i^\dagger \tilde{\mathbf{R}} \tilde{\gamma}_j$  for all  $i, j$ . Using standard results regarding the equivalence of eigenvector sets of a hermitian

(5) Let  $z_1, z_2, z_3, z_4$  be jointly circular Gaussian random variables with zero mean. Then [15]

$$E[z_1 z_2^* z_3^* z_4] = E[z_1 z_2^*] E[z_3 z_4^*] + E[z_1 z_3^*] E[z_2 z_4^*].$$

matrix (see footnote (4)), it then follows that  $\tilde{\gamma}_i = \tilde{\beta}_i e^{j\psi_i}$ ,  $i = 1, 2, \dots, K$  and  $[\tilde{\gamma}_{K+1}, \dots, \tilde{\gamma}_M] = [\tilde{\beta}_{K+1}, \dots, \tilde{\beta}_M] \mathbf{V}$ , where  $\mathbf{V}$  is an  $(M-K) \times (M-K)$  unitary matrix. Further,

$$\mathbf{R}_{pq}^f \triangleq E[\mathbf{x}_p^f (\mathbf{x}_q^f)^\dagger] ; \mathbf{R}_{pq}^b \triangleq E[\mathbf{x}_p^b (\mathbf{x}_q^b)^\dagger] \quad (\text{A.17})$$

In obtaining (A.16), we have also made use of (A.15) and the fact that for circular Gaussian data,

$$E[z_{pi}^f z_{qj}^{b*}] = \tilde{\beta}_i^\dagger E[\mathbf{x}_p^f \mathbf{x}_q^{b*}] \tilde{\beta}_j = \tilde{\beta}_i^\dagger E[\mathbf{x}_p^f (\mathbf{x}_q^f)^T] \gamma_j^* = 0, \quad p, q = 1 \rightarrow L.$$

The forward-only smoothing scheme now follows as a subclass of this general analysis. In that case the estimate of the smoothed covariance matrix is given by

$$\mathbf{S}^f = \frac{1}{L} \sum_{l=1}^L \left[ \frac{1}{N} \sum_{n=1}^N \mathbf{x}_l^f(n) (\mathbf{x}_l^f(n))^\dagger \right] = \mathbf{E}_f \mathbf{L}_f \mathbf{E}_f^\dagger.$$

Then, as in (A.14) with  $z_{li}^f(n) = \beta_{fi}^\dagger \mathbf{x}_l^f(n)$  where  $E[\mathbf{S}^f] = \mathbf{R}^f = \mathbf{B}_f \mathbf{A}_f \mathbf{B}_f^\dagger = \sum_{i=1}^M \lambda_{fi} \beta_{fi} \beta_{fi}^\dagger$ , we have

$$u_{ij} = \frac{1}{\sqrt{N}} \sum_{n=1}^N \left[ \frac{1}{L} \sum_{l=1}^L z_{li}^f z_{lj}^{f*} - \lambda_{fi} \delta_{ij} \right].$$

Thus,  $E[u_{ij}] = 0$  and

$$E[u_{ij} u_{kl}^*] = \frac{1}{L^2} \sum_{p=1}^L \sum_{q=1}^L \beta_{fi}^\dagger \mathbf{R}_{pq}^f \beta_{fk} \beta_{fl}^\dagger \mathbf{R}_{qp}^f \beta_{fj} \triangleq \Gamma_{iklj}^f \quad (\text{A.18})$$

where we have again made use of the circular Gaussian property of the data vectors.

Two important special cases of considerable practical interest are the conventional (unsmoothed) MUSIC scheme and the f/b smoothed scheme with  $L=1$ . Of these, the former one corresponds to the forward-only scheme with  $L=1$ , and in that case from (A.18),

$$\Gamma_{iklj} = \beta_i^\dagger \mathbf{R} \beta_k \beta_l^\dagger \mathbf{R} \beta_j = \lambda_i \lambda_j \delta_{ik} \delta_{jl}. \quad (\text{A.19})$$

In the latter case, using (A.17) together with the identity  $\tilde{\beta}_i^\dagger \mathbf{a}(\omega_k) = 0$  for  $i = K+1, K+2, \dots, M$  and  $k = 1, 2, \dots, K$  simplifies (A.16) into

$$\tilde{\Gamma}_{iklj} = \frac{1}{2} \tilde{\lambda}_i \tilde{\lambda}_j (\delta_{ik} \delta_{jl} + \tilde{\beta}_i^\dagger \tilde{\gamma}_l \tilde{\gamma}_j^\dagger \tilde{\beta}_k)$$

in an uncorrelated scene. Similarly, in a two equipowered coherent source scene, we also have

$$\tilde{\Gamma}_{iklj} = \begin{cases} \tilde{\lambda}_i \tilde{\lambda}_j \delta_{ik} \delta_{jl} - \frac{1}{4} \left[ \tilde{\beta}_i^\dagger \mathbf{R}_{11}^f \tilde{\beta}_k \tilde{\beta}_l^\dagger \mathbf{R}_{11}^b \tilde{\beta}_j + \tilde{\beta}_i^\dagger \mathbf{R}_{11}^b \tilde{\beta}_k \tilde{\beta}_l^\dagger \mathbf{R}_{11}^f \tilde{\beta}_j \right. \\ \left. - \tilde{\beta}_i^\dagger \mathbf{R}_{11}^f \tilde{\gamma}_l \tilde{\gamma}_j^\dagger \mathbf{R}_{11}^b \tilde{\beta}_k - \tilde{\beta}_i^\dagger \mathbf{R}_{11}^b \tilde{\gamma}_l \tilde{\gamma}_j^\dagger \mathbf{R}_{11}^f \tilde{\beta}_k \right], & i, j, k, l \leq 2 \\ \frac{1}{2} \tilde{\lambda}_i \tilde{\lambda}_j (\delta_{ik} \delta_{jl} + \tilde{\beta}_i^\dagger \tilde{\gamma}_l \tilde{\gamma}_j^\dagger \tilde{\beta}_k), & \text{otherwise.} \end{cases} \quad (\text{A.20})$$

Further, using (A.11) together with (A.12) and (A.9) we have

$$\mathbf{T} = \tilde{\mathbf{A}} + \frac{1}{\sqrt{N}} \mathbf{U} = \mathbf{Y} \tilde{\mathbf{L}} \mathbf{Y}^\dagger = \mathbf{Y} (\tilde{\mathbf{A}} + \frac{1}{\sqrt{N}} \mathbf{F}) \mathbf{Y}^\dagger$$

which gives the useful identity

$$\tilde{\mathbf{A}} + \frac{1}{\sqrt{N}} \mathbf{U} = \mathbf{Y} (\tilde{\mathbf{A}} + \frac{1}{\sqrt{N}} \mathbf{F}) \mathbf{Y}^\dagger. \quad (\text{A.21})$$

To derive the asymptotic properties of the sample estimates corresponding to the distinct eigenvalues  $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_K$  of  $\tilde{\mathbf{R}}$ , following [16, 19] we partition the matrices  $\tilde{\mathbf{A}}$ ,  $\mathbf{U}$ ,  $\mathbf{F}$  and  $\mathbf{Y}$  as follows

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{A}}_1 & \mathbf{O} \\ \mathbf{O} & \sigma^2 \mathbf{I}_{M-K} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{F}_2 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} \end{bmatrix}. \quad (\text{A.22})$$

Here  $\tilde{\mathbf{A}}_1$ ,  $\mathbf{U}_{11}$ ,  $\mathbf{F}_1$  and  $\mathbf{Y}_{11}$  are of sizes  $K \times K$ , etc. With (A.22) in (A.21) and after some algebraic manipulations and retaining only those terms of order less than or equal to  $1/\sqrt{N}$ , we have

$$\begin{aligned} & \begin{bmatrix} \tilde{\mathbf{A}}_1 & \mathbf{O} \\ \mathbf{O} & \sigma^2 \mathbf{I}_{M-K} \end{bmatrix} + \frac{1}{\sqrt{N}} \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{A}}_1 & \mathbf{O} \\ \mathbf{O} & \sigma^2 \mathbf{Y}_{22} \mathbf{Y}_{22}^\dagger \end{bmatrix} \\ & + \frac{1}{\sqrt{N}} \left( \begin{bmatrix} \tilde{\mathbf{A}}_1 \mathbf{W}_{11}^\dagger & \tilde{\mathbf{A}}_1 \mathbf{W}_{21}^\dagger \\ \sigma^2 \mathbf{Y}_{22} \mathbf{W}_{12}^\dagger & \mathbf{O} \end{bmatrix} + \begin{bmatrix} \mathbf{F}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{Y}_{22} \mathbf{F}_2 \mathbf{Y}_{22}^\dagger \end{bmatrix} + \begin{bmatrix} \mathbf{W}_{11} \tilde{\mathbf{A}}_1 & \sigma^2 \mathbf{W}_{12} \mathbf{Y}_{22}^\dagger \\ \mathbf{W}_{21} \tilde{\mathbf{A}}_1 & \mathbf{O} \end{bmatrix} \right) \\ & + o\left(\frac{1}{N}\right) \end{aligned} \quad (\text{A.23})$$

where

$$\mathbf{W}_{11} = \sqrt{N} (\mathbf{Y}_{11} - \mathbf{I}_K) \quad (\text{A.24})$$

$$\mathbf{W}_{12} = \sqrt{N} \mathbf{Y}_{12}, \quad \mathbf{W}_{21} = \sqrt{N} \mathbf{Y}_{21}$$

and define the column vectors  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_K$  from

$$\begin{bmatrix} \mathbf{W}_{11} \\ \mathbf{W}_{21} \end{bmatrix} = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_K] \triangleq \mathbf{W}.$$

Similarly,

$$\begin{aligned} \mathbf{Y} \mathbf{Y}^\dagger = \mathbf{I}_M &= \begin{bmatrix} \mathbf{I}_K & \mathbf{O} \\ \mathbf{O} & \mathbf{Y}_{22} \mathbf{Y}_{22}^\dagger \end{bmatrix} + \\ & \frac{1}{\sqrt{N}} \left( \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \mathbf{Y}_{22}^\dagger \\ \mathbf{W}_{21} & \mathbf{O} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_{11}^\dagger & \mathbf{W}_{21}^\dagger \\ \mathbf{Y}_{22} \mathbf{W}_{12}^\dagger & \mathbf{O} \end{bmatrix} \right) + o\left(\frac{1}{N}\right). \end{aligned} \quad (\text{A.25})$$

Thus, asymptotically for sufficiently large  $N$ , from (A.25) and (A.23) we have

$$\mathbf{O} = \mathbf{W}_{11} + \mathbf{W}_{11}^\dagger \quad (\text{A.26})$$

$$\mathbf{W}_{21} + \mathbf{Y}_{22} \mathbf{W}_{12}^\dagger = \mathbf{O}$$

$$\mathbf{U}_{11} = \mathbf{W}_{11} \tilde{\mathbf{A}}_1 + \mathbf{F}_1 + \tilde{\mathbf{A}}_1 \mathbf{W}_{11}^\dagger \quad (\text{A.27})$$

and

$$\mathbf{U}_{21} = \mathbf{W}_{21} \tilde{\mathbf{A}}_1 + \sigma^2 \mathbf{Y}_{22} \mathbf{W}_{12}^\dagger = \mathbf{W}_{21} \tilde{\mathbf{A}}_1 - \sigma^2 \mathbf{W}_{21}. \quad (\text{A.28})$$

Since  $y_{ii} \geq 0$ , this together with (A.24) and (A.26) implies

$$w_{ii} = 0, \quad i = 1, 2, \dots, K$$

and

$$w_{ij} = -w_{ji}^*, \quad i, j = 1, 2, \dots, K, \quad i \neq j$$

which when substituted into (A.27) - (A.28) gives

$$f_{ii} = u_{ii}, \quad i = 1, 2, \dots, K \quad (\text{A.29})$$

$$w_{ij} = \begin{cases} u_{ij}/(\tilde{\lambda}_j - \tilde{\lambda}_i) & i, j = 1, 2, \dots, K, \quad i \neq j \\ u_{ij}/(\tilde{\lambda}_j - \sigma^2) & i = K+1, K+2, \dots, M, \quad j = 1, 2, \dots, K. \end{cases} \quad (\text{A.30})$$

From (A.10) and (A.5) we also have

$$\mathbf{G} = \sqrt{N} (\mathbf{C} - \tilde{\mathbf{B}}) = \sqrt{N} \tilde{\mathbf{B}}(\mathbf{Y} - \mathbf{I}) = \sqrt{N} \begin{bmatrix} \mathbf{Y}_{11} - \mathbf{I}_K & \mathbf{Y}_{12} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} - \mathbf{I}_{M-K} \end{bmatrix}$$

which gives

$$[\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_K] = \tilde{\mathbf{B}}[\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_K].$$

This together with (A.9) and (A.29) gives

$$\tilde{l}_i = \tilde{\lambda}_i + (1/\sqrt{N}) f_{ii} = \tilde{\lambda}_i + (1/\sqrt{N}) u_{ii}; \quad i = 1, 2, \dots, K \quad (\text{A.31})$$

and

$$\mathbf{c}_i = \tilde{\boldsymbol{\beta}}_i + (1/\sqrt{N}) \tilde{\mathbf{B}} \mathbf{w}_i = \tilde{\boldsymbol{\beta}}_i + (1/\sqrt{N}) \sum_{\substack{j=1 \\ j \neq i}}^M w_{ji} \tilde{\boldsymbol{\beta}}_j; \quad i = 1, 2, \dots, K. \quad (\text{A.32})$$

Thus, the estimators  $\tilde{l}_i$  and  $\mathbf{c}_i$ ,  $i = 1, 2, \dots, K$ , corresponding to the distinct eigenvalues of  $\tilde{\mathbf{R}}$ , are asymptotically multivariate Gaussian random variables/vectors with

mean values  $\bar{\lambda}_i$  and  $\bar{\beta}_i$ ,  $i = 1, 2, \dots, K$ , respectively. Further,

$$\text{Cov}(\bar{l}_i, \bar{l}_j) = \frac{1}{N} E[u_{ii} u_{jj}] = \frac{1}{N} \bar{\Gamma}_{ijji}, \quad i, j = 1, 2, \dots, K \quad (\text{A.33})$$

and

$$\text{Cov}(\mathbf{c}_i, \mathbf{c}_j) = \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq j}}^M \frac{\bar{\Gamma}_{klji}}{(\bar{\lambda}_i - \bar{\lambda}_k)(\bar{\lambda}_j - \bar{\lambda}_l)} \bar{\beta}_k \bar{\beta}_l^\dagger. \quad (\text{A.34})$$

Notice that  $\mathbf{c}_i$  in (A.32) are not normalized vectors, and it may be emphasized that in the case of eigenvectors, the above asymptotic joint Gaussian property only holds good for these specific sets of unnormalized sample estimators. However, from (A.4) since the eigenvectors  $\mathbf{c}_i$ ,  $i = 1, 2, \dots, K$ , appearing in (A.7) are normalized ones, to make use of the explicit forms given by (A.32) there, we proceed to normalize these vectors. Starting from (A.32), we have

$$\|\mathbf{c}_i\|^2 = \mathbf{c}_i^\dagger \mathbf{c}_i = 1 + \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^M |w_{ji}|^2 > 1$$

and, hence, the corresponding normalized eigenvectors  $\hat{\mathbf{c}}_i$ ,  $i = 1, 2, \dots, K$  have the form

$$\begin{aligned} \hat{\mathbf{c}}_i \triangleq \|\mathbf{c}_i\|^{-1} \mathbf{c}_i &\approx \left[ 1 - \frac{1}{2N} \sum_{\substack{j=1 \\ j \neq i}}^M |w_{ji}|^2 \right] \bar{\beta}_i + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^M w_{ji} \bar{\beta}_j \\ &\quad - \frac{1}{2N\sqrt{N}} \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq i}}^M |w_{ki}|^2 w_{li} \bar{\beta}_l + o(1/N^2). \end{aligned}$$

Using (A.30) and (A.16) this gives

$$E[\hat{\mathbf{c}}_i] = \bar{\beta}_i - \frac{1}{2N} \sum_{\substack{j=1 \\ j \neq i}}^M \frac{\bar{\Gamma}_{iijj}}{(\bar{\lambda}_i - \bar{\lambda}_j)^2} \bar{\beta}_i + o(1/N^2); \quad i = 1, 2, \dots, K \quad (\text{A.35})$$

since from the asymptotic joint normal distribution of these zero-mean random variables  $u_{ij}$  ( $i \neq j$ ), their odd-order moments are zero. Thus, asymptotically these normalized estimates for the eigenvectors  $\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_K$ , of  $\bar{\mathbf{R}}$  are unbiased and the exact bias expressions are given by (A.35). Further,

$$\begin{aligned}
 \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j^\dagger &= \left[ 1 - \frac{1}{2N} \left( \sum_{\substack{k=1 \\ k \neq i}}^M |w_{ki}|^2 + \sum_{\substack{k=1 \\ k \neq j}}^M |w_{kj}|^2 \right) \right] \tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_j^\dagger \\
 &+ \frac{1}{\sqrt{N}} \left( \sum_{\substack{k=1 \\ k \neq i}}^M w_{ki} \tilde{\mathbf{p}}_k \tilde{\mathbf{p}}_j^\dagger + \sum_{\substack{k=1 \\ k \neq j}}^M w_{kj}^* \tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_k^\dagger \right) + \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq j}}^M w_{ki} w_{lj}^* \tilde{\mathbf{p}}_k \tilde{\mathbf{p}}_l^\dagger \\
 &- \frac{1}{2N\sqrt{N}} \left( \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq i}}^M |w_{ki}|^2 w_{li} \tilde{\mathbf{p}}_l \tilde{\mathbf{p}}_j^\dagger + \sum_{\substack{k=1 \\ k \neq j}}^M \sum_{\substack{l=1 \\ l \neq j}}^M |w_{kj}|^2 w_{lj}^* \tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_l^\dagger \right. \\
 &\left. + \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq j}}^M w_{ki} |w_{lj}|^2 \tilde{\mathbf{p}}_k \tilde{\mathbf{p}}_j^\dagger + \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq j}}^M |w_{ki}|^2 w_{lj}^* \tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_l^\dagger \right) + o(1/N^2). \quad (\text{A.36})
 \end{aligned}$$

Again, neglecting terms of order  $1/N^2$  and proceeding as above, this expression reduces to

$$\begin{aligned}
 E[\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j^\dagger] &= \tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_j^\dagger - \frac{1}{2N} \left[ \sum_{\substack{k=1 \\ k \neq i}}^M \frac{\bar{\Gamma}_{iikk}}{(\bar{\lambda}_i - \bar{\lambda}_k)^2} + \sum_{\substack{k=1 \\ k \neq j}}^M \frac{\bar{\Gamma}_{jjkk}}{(\bar{\lambda}_j - \bar{\lambda}_k)^2} \right] \tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_j^\dagger \\
 &+ \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq j}}^M \frac{\bar{\Gamma}_{klji}}{(\bar{\lambda}_i - \bar{\lambda}_k)(\bar{\lambda}_j - \bar{\lambda}_l)} \tilde{\mathbf{p}}_k \tilde{\mathbf{p}}_l^\dagger + o(1/N^2) \quad (\text{A.37})
 \end{aligned}$$

An easy verification shows that  $\text{Cov}(\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j)$  is once again given by (A.34), but nevertheless, (A.36) - (A.37) will turn out to be useful in computing the asymptotic bias and variance of the sample direction estimator  $\hat{\mathbf{Q}}(\omega)$  in (21).

These general expressions in (A.35) - (A.37) for first- and second-order moments of normalized eigenvector estimators can be used to evaluate their counterparts for the forward-only and the conventional cases by substituting the proper  $\Gamma$  values derived in (A.18) and (A.19), respectively. Thus, for the conventional MUSIC case



using (A.19), we have

$$E[\hat{c}_i] = \beta_i - \frac{1}{2N} \sum_{\substack{j=1 \\ j \neq i}}^M \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \beta_i + o(1/N^2)$$

$$E[\hat{c}_i \hat{c}_j^\dagger] = \beta_i \beta_j^\dagger - \frac{1}{2N} \left[ \sum_{\substack{k=1 \\ k \neq i}}^M \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} + \sum_{\substack{k=1 \\ k \neq j}}^M \frac{\lambda_j \lambda_k}{(\lambda_j - \lambda_k)^2} \right] \beta_i \beta_j^\dagger$$

$$+ \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^M \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} \beta_k \beta_k^\dagger \delta_{ij} + o(1/N^2)$$

where  $\lambda_i, \beta_i$ ;  $i = 1, 2, \dots, M$  are as defined in (8). These results coincide with those in [13] (see Appendix A).

Once again for the f/b smoothed case using (A.7) - (A.8) and recalling that the eigenvector estimator appearing there are normalized ones, we have

$$y_i(\omega) = |\hat{c}_i^\dagger \mathbf{a}(\omega)|^2 = \mathbf{a}^\dagger(\omega) \hat{c}_i \hat{c}_i^\dagger \mathbf{a}(\omega) \quad (\text{A.38})$$

and from (A.37) and (21) we have

$$E[\hat{Q}(\omega)] = \bar{Q}(\omega) + \frac{1}{N} \sum_{i=1}^K \left[ \sum_{\substack{k=1 \\ k \neq i}}^M \frac{\bar{\Gamma}_{iikk}}{(\bar{\lambda}_i - \bar{\lambda}_k)^2} |\bar{\beta}_i^\dagger \mathbf{a}(\omega)|^2 \right.$$

$$\left. - \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq i}}^M \frac{\bar{\Gamma}_{iikl}}{(\bar{\lambda}_i - \bar{\lambda}_k)(\bar{\lambda}_i - \bar{\lambda}_l)} \mathbf{a}^\dagger(\omega) \bar{\beta}_k \bar{\beta}_l^\dagger \mathbf{a}(\omega) \right] + o(1/N^2) \rightarrow \bar{Q}(\omega) \text{ as } N \rightarrow \infty. \quad (\text{A.39})$$

Also,

$$\text{Var}(\hat{Q}(\omega)) = E[\hat{Q}^2(\omega)] - \left(E[\hat{Q}(\omega)]\right)^2 = \sum_{i=1}^K \sum_{j=1}^K \left(E[y_i y_j] - E[y_i] E[y_j]\right). \quad (\text{A.40})$$

Using (A.38) and (A.36), after a series of manipulations, we have

$$E[y_i y_j] - E[y_i]E[y_j] = \frac{1}{N} E[d_i(\omega) d_j(\omega)] + o(1/N^2)$$

where

$$d_i(\omega) = \sum_{\substack{k=1 \\ k \neq i}}^M \left( w_{ki} a^\dagger(\omega) \tilde{\beta}_k \tilde{\beta}_i^\dagger a(\omega) + w_{ki}^* a^\dagger(\omega) \tilde{\beta}_i \tilde{\beta}_k^\dagger a(\omega) \right)$$

which gives

$$E[y_i y_j] - E[y_i]E[y_j] = \frac{2}{N} \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq j}}^M \frac{\text{Re} \left[ \left( \tilde{\Gamma}_{klji} a^\dagger(\omega) \tilde{\beta}_j \tilde{\beta}_l^\dagger a(\omega) + \tilde{\Gamma}_{kjli} a^\dagger(\omega) \tilde{\beta}_l \tilde{\beta}_j^\dagger a(\omega) \right) a^\dagger(\omega) \tilde{\beta}_k \tilde{\beta}_i^\dagger a(\omega) \right]}{(\tilde{\lambda}_i - \tilde{\lambda}_k)(\tilde{\lambda}_j - \tilde{\lambda}_l)} + o\left(\frac{1}{N^2}\right). \quad (\text{A.41})$$

Finally, with (A.41) in (A.40) we get

$$\begin{aligned} \text{Var}(\hat{Q}(\omega)) = & \frac{2}{N} \sum_{i=1}^K \sum_{j=1}^K \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\substack{l=1 \\ l \neq j}}^M \frac{\text{Re} \left[ \left( \tilde{\Gamma}_{klji} a^\dagger(\omega) \tilde{\beta}_j \tilde{\beta}_l^\dagger a(\omega) + \tilde{\Gamma}_{kjli} a^\dagger(\omega) \tilde{\beta}_l \tilde{\beta}_j^\dagger a(\omega) \right) a^\dagger(\omega) \tilde{\beta}_k \tilde{\beta}_i^\dagger a(\omega) \right]}{(\tilde{\lambda}_i - \tilde{\lambda}_k)(\tilde{\lambda}_j - \tilde{\lambda}_l)} \\ & + o(1/N^2) \rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned} \quad (\text{A.42})$$

Thus  $\hat{Q}(\omega)$  is a consistent estimator in all cases.

## Appendix B

In this appendix, expressions for eigenvalues and eigenvectors of the smoothed covariance matrix  $\hat{\mathbf{R}}$  with  $L = 1$  for two coherent signals are derived. In addition, several associated inner products that are needed in section 2.2.3.B for resolution performance evaluation are also developed [20].

Consider two coherent sources  $\alpha_1 u(t)$  and  $\alpha_2 u(t)$  with arrival angles  $\theta_1, \theta_2$  and source covariance matrix

$$\begin{bmatrix} |\alpha_1|^2 & \alpha_1 \alpha_2^* \\ \alpha_2 \alpha_1^* & |\alpha_2|^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} \alpha_1^* & \alpha_2^* \end{bmatrix} \triangleq \boldsymbol{\alpha} \boldsymbol{\alpha}^\dagger, \quad \alpha_i = |\alpha_i| e^{j\phi_i}, \quad E[|u(t)|^2] = 1.$$

When the forward/backward smoothing scheme is deployed once to decorrelate the coherent signals, from (15) the resulting source covariance matrix  $\tilde{\mathbf{R}}_u$  has the form

$$\tilde{\mathbf{R}}_u = \frac{1}{2} [\boldsymbol{\alpha} \boldsymbol{\alpha}^\dagger + \boldsymbol{\gamma} \boldsymbol{\gamma}^\dagger] = \begin{bmatrix} |\alpha_1|^2 & \alpha_1 \alpha_2^* \rho_t \\ \alpha_2 \alpha_1^* \rho_t^* & |\alpha_2|^2 \end{bmatrix} = \frac{1}{2} [\boldsymbol{\alpha} \boldsymbol{\gamma}] \begin{bmatrix} \boldsymbol{\alpha}^\dagger \\ \boldsymbol{\gamma}^\dagger \end{bmatrix} \quad (\text{B.1})$$

where

$$\boldsymbol{\gamma} = [\nu_1 \alpha_1^* \quad \nu_2 \alpha_2^*]^T, \quad \nu_1 = e^{j(M-1)\omega_1}, \quad \nu_2 = e^{j(M-1)\omega_2}$$

and the effective correlation coefficient  $\rho_t$  is given by

$$\rho_t = (1 + e^{-j2(\phi_1 - \phi_2)} \nu_1 \nu_2^*) / 2 \quad (\text{B.2})$$

Using (B.1), the noiseless part of the smoothed covariance matrix  $\tilde{\mathbf{R}}$  can be written as

$$\tilde{\mathbf{R}} \triangleq \mathbf{A} \tilde{\mathbf{R}}_u \mathbf{A}^\dagger = \frac{1}{2} [\mathbf{b}_1 \mathbf{b}_1^\dagger + \mathbf{b}_2 \mathbf{b}_2^\dagger] \quad (\text{B.3})$$

where

$$\mathbf{b}_1 = \sqrt{M} (\alpha_1 \mathbf{a}(\omega_1) + \alpha_2 \mathbf{a}(\omega_2)); \quad \mathbf{b}_2 = \sqrt{M} (\nu_1 \alpha_1^* \mathbf{a}(\omega_1) + \nu_2 \alpha_2^* \mathbf{a}(\omega_2)). \quad (\text{B.4})$$

The nonzero eigenvalues of  $\tilde{\mathbf{R}}$  are given by the roots of the quadratic equation

$$\tilde{\mu}^2 - \text{tr}(\mathbf{A} \tilde{\mathbf{R}}_u \mathbf{A}^\dagger) \tilde{\mu} + |\tilde{\mathbf{R}}_u \mathbf{A}^\dagger \mathbf{A}| = 0$$

which gives

$$\tilde{\mu}_i = \frac{1}{2} \text{tr}(\mathbf{A} \tilde{\mathbf{R}}_u \mathbf{A}^\dagger) \left[ 1 \pm \sqrt{1 - \frac{|\tilde{\mathbf{R}}_u \mathbf{A}^\dagger \mathbf{A}|}{[\text{tr}(\mathbf{A} \tilde{\mathbf{R}}_u \mathbf{A}^\dagger)/2]^2}} \right]^{1/2}; \quad i = 1, 2. \quad (\text{B.5})$$

To evaluate  $\tilde{\mu}_1, \tilde{\mu}_2$  explicitly, we define the spatial correlation coefficient  $\rho_s$  between the sources to be

$$\rho_s = \mathbf{a}(\omega_1)^\dagger \mathbf{a}(\omega_2) = e^{j(M-1)\omega_d} \frac{\sin M\omega_d}{M \sin \omega_d} \triangleq e^{j(M-1)\omega_d} \text{Si}(M\omega_d) \quad (\text{B.6})$$

with  $\omega_d = (\omega_1 - \omega_2)/2$ . From (B.3) - (B.4) we have

$$\text{tr}(\mathbf{A}\tilde{\mathbf{R}}_u\mathbf{A}^\dagger) = M (|\alpha_1|^2 + |\alpha_2|^2 + \text{Re}(\alpha_1^*\alpha_2\rho_s + \alpha_1\alpha_2^*\nu_1^*\nu_2\rho_s)) \quad (\text{B.7})$$

and

$$|\tilde{\mathbf{R}}_u\mathbf{A}^\dagger\mathbf{A}| = |\tilde{\mathbf{R}}_u| |\mathbf{A}^\dagger\mathbf{A}| = M^2 |\alpha_1|^2 |\alpha_2|^2 (1 - |\rho_t|^2)(1 - |\rho_s|^2). \quad (\text{B.8})$$

Here onwards we consider the two sources to be perfectly coherent and of equal power; i.e.,  $\alpha_1 = \alpha_2$  and  $|\alpha_1|^2 = 1$ . In that case we have

$$\rho_t = \frac{1 + \nu_1\nu_2^*}{2} = e^{j(M-1)\omega_d} \cos(M-1)\omega_d \quad (\text{B.9})$$

$$\text{Re}(\rho_s) = \cos[(M-1)\omega_d] \text{Si}(M\omega_d) = \text{Re}(\rho_s\rho_t^*) \quad (\text{B.10})$$

and with (B.7) - (B.8) in (B.5) it is easy to see that

$$\tilde{\mu}_i = M \left[ 1 + \text{Re}(\rho_s\rho_t^*) \pm |\rho_s + \rho_t| \right]; \quad i = 1, 2, \quad (\text{B.11})$$

where we have made use of the identity  $[\text{Re}(\rho_s\rho_t^*)]^2 = |\rho_s|^2 |\rho_t|^2$ . The eigenvectors corresponding to these eigenvalues span the two-dimensional signal subspace, and from (B.3) they are linear combinations of  $\mathbf{b}_1$  and  $\mathbf{b}_2$ ; i.e.,

$$\tilde{\beta}_i \propto (\mathbf{b}_1 + k_i \mathbf{b}_2), \quad i = 1, 2. \quad (\text{B.12})$$

Moreover,  $\tilde{\mu}_i, \tilde{\beta}_i, i = 1, 2$ , as a pair satisfy

$$(\mathbf{A}\tilde{\mathbf{R}}_u\mathbf{A}^\dagger)\tilde{\beta}_i = \tilde{\mu}_i\tilde{\beta}_i, \quad i = 1, 2,$$

which together with (B.3) results in

$$\frac{1}{2} [\mathbf{b}_1^\dagger\mathbf{b}_1 + k_i\mathbf{b}_1^\dagger\mathbf{b}_2] \mathbf{b}_1 + \frac{1}{2} [\mathbf{b}_2^\dagger\mathbf{b}_1 + k_i\mathbf{b}_2^\dagger\mathbf{b}_2] \mathbf{b}_2 = \tilde{\mu}_i \mathbf{b}_1 + \tilde{\mu}_i k_i \mathbf{b}_2, \quad i = 1, 2. \quad (\text{B.13})$$

The solution to (B.13) need not be unique. However since the eigenvectors can be made unique by proper normalization, at this stage we seek a solution set to the distinct equations

$$\frac{1}{2} [\mathbf{b}_1^\dagger\mathbf{b}_1 + k_i\mathbf{b}_1^\dagger\mathbf{b}_2] = \tilde{\mu}_i; \quad \frac{1}{2} [\mathbf{b}_2^\dagger\mathbf{b}_1 + k_i\mathbf{b}_2^\dagger\mathbf{b}_2] = \tilde{\mu}_i k_i, \quad i = 1, 2. \quad (\text{B.14})$$

Clearly, solutions to (B.14), if they exist, satisfy (B.13). To simplify  $k_i$  ;  $i = 1, 2$  further, using (B.4) and (B.10), notice that

$$\mathbf{b}_1^\dagger \mathbf{b}_1 = 2M (1 + \text{Re}(\rho_s \rho_t^*))$$

$$\mathbf{b}_1^\dagger \mathbf{b}_2 = 2\alpha_1^{*2} M \nu_1(\rho_s^* + \rho_t^*) = 2\alpha_1^{*2} M \nu_2(\rho_s + \rho_t)$$

and these together with (B.11) in (B.14) yield

$$k_i = \pm \frac{|\rho_s + \rho_t|}{\alpha_1^{*2} \nu_1(\rho_s^* + \rho_t^*)} = \pm \frac{|\rho_s + \rho_t|}{\alpha_1^{*2} \nu_2(\rho_s + \rho_t)}. \quad (\text{B.15})$$

Using (B.15), the eigenvectors in (B.12) can be written as

$$\begin{aligned} \tilde{\beta}_i &\propto (\mathbf{b}_1 + k_i \mathbf{b}_2) = (\alpha_1 + \alpha_1^* \nu_1 k_i) \mathbf{a}(\omega_1) + (\alpha + \alpha_1^* \nu_2 k_i) \mathbf{a}(\omega_2) \\ &\propto \left[ 1 \pm \frac{|\rho_s + \rho_t|}{\rho_s^* + \rho_t^*} \right] \mathbf{a}(\omega_1) + \left[ 1 \pm \frac{|\rho_s + \rho_t|}{\rho_s + \rho_t} \right] \mathbf{a}(\omega_2). \end{aligned} \quad (\text{B.16})$$

To simplify this further observe that

$$\rho_s + \rho_t = e^{j(M-1)\omega_d} [\cos(M-1)\omega_d + Si(M\omega_d)]$$

and consider the case when  $[\cos(M-1)\omega_d + Si(M\omega_d)] > 0$ . Then from (B.16)

$$\begin{aligned} \tilde{\beta}_i &\propto e^{j(M-1)\omega_1/2} \left[ e^{-j(M-1)\omega_1/2} \pm e^{-j(M-1)\omega_2/2} \right] \mathbf{a}(\omega_1) \\ &\pm e^{j(M-1)\omega_2/2} \left[ e^{-j(M-1)\omega_1/2} \pm e^{-j(M-1)\omega_2/2} \right] \mathbf{a}(\omega_2) \propto \mathbf{u}_1 \pm \mathbf{u}_2 \end{aligned}$$

where

$$\mathbf{u}_1 = e^{j(M-1)\omega_1/2} \mathbf{a}(\omega_1), \quad \mathbf{u}_2 = e^{j(M-1)\omega_2/2} \mathbf{a}(\omega_2).$$

Finally, the normalized eigenvectors corresponding to the nonzero eigenvalues in (B.11) are given by

$$\tilde{\beta}_i = \begin{cases} (\mathbf{u}_1 \pm \mathbf{u}_2) / \sqrt{2(1 \pm Si(M\omega_d))} & [\cos(M-1)\omega_d + Si(M\omega_d)] > 0 \\ (\mathbf{u}_1 \mp \mathbf{u}_2) / \sqrt{2(1 \mp Si(M\omega_d))} & \text{otherwise} \end{cases} \quad (B.17)$$

$$i = 1, 2.$$

Notice that the uncorrelated source scene is a special case of (B.1) with  $\alpha_1 = \alpha_2 = 1$ , and  $\rho_t = 0$ . Thus from (B.11) in that case

$$\mu_i = M(1 \pm |\rho_s|) \quad (B.18)$$

and similarly

$$\beta_i = \begin{cases} (\mathbf{u}_1 \pm \mathbf{u}_2) / \sqrt{2(1 \pm Si(M\omega_d))} & Si(M\omega_d) > 0 \\ (\mathbf{u}_1 \mp \mathbf{u}_2) / \sqrt{2(1 \mp Si(M\omega_d))} & \text{otherwise} \end{cases} \quad (B.19)$$

$$i = 1, 2.$$

From (B.17) and (B.19) we can conclude that for equipowered sources, irrespective of their effective correlation  $\rho_t$  resulting from spatial smoothing, the smoothed and the uncorrelated cases have the same set of eigenvectors whenever  $[\cos(M-1)\omega_d + Si(M\omega_d)]$  and  $Si(M\omega_d)$  have the same sign; i.e.,

$$\tilde{\beta}_i = \beta_i, i = 1, 2 \quad [\cos(M-1)\omega_d + Si(M\omega_d)] \gtrless 0 \text{ and } Si(M\omega_d) \gtrless 0$$

$$\tilde{\beta}_1 = \beta_2, \tilde{\beta}_2 = \beta_1 \quad \text{otherwise}.$$

We conclude this appendix with several useful parametric approximations to eigenvalues and inner products between eigenvectors and direction vectors for both uncorrelated and coherent cases. To start with, let [13]

$$\Delta^2 \triangleq (M^2 \omega_d^2) / 3$$

For closely spaced sources  $(M\omega_d)^2 \ll 1$ , and in that case from (B.9) and (B.6)

$$\rho_t = e^{j(M-1)\omega_d} \left[ 1 - \frac{3}{2} \left( \frac{M-1}{M} \right)^2 \Delta^2 + \frac{3}{8} \left( \frac{M-1}{M} \right)^4 \Delta^4 + \dots \right]$$

and

$$Si(M\omega_d) = \frac{1}{M} \frac{\sin M\omega_d}{\sin \omega_d} \approx [1 - \frac{1}{2}\Delta^2 + \frac{3}{40}\Delta^4]$$

$$\rho_s \approx e^{j(M-1)\omega_d} \left[ 1 - \frac{1}{2}\Delta^2 + \frac{3}{40}\Delta^4 \right]. \quad (B.20)$$

For the uncorrelated case parametric expressions for the eigenvalues can be easily obtained from (B.18) and (B.20). In a similar manner for the coherent case, using (B.11) we get

$$\tilde{\mu}_1 \approx 4M \left[ 1 - \left(1 - \frac{3}{2M}\right)\Delta^2 + \left(\frac{33}{80} - \frac{9}{8M}\right)\Delta^4 \right] \quad (B.21)$$

$$\tilde{\mu}_2 \approx \frac{3}{4}M \left(1 - \frac{2}{M}\right)\Delta^4. \quad (B.22)$$

Because of the equivalence of eigenvectors for two uncorrelated and perfectly coherent equipowered sources, we have (for  $Si(M\omega_d) > 0$ )

$$|\tilde{\beta}_i^\dagger \mathbf{a}(\omega)|^2 = |\beta_i^\dagger \mathbf{a}(\omega)|^2, \text{ for all } \omega, i = 1, 2 \quad (B.23)$$

and it suffices to obtain the inner products of the eigenvectors in (B.19) with direction vectors associated with the true arrival angles. A little algebra shows that

$$|\tilde{\beta}_1^\dagger \mathbf{a}(\omega_1)|^2 = |\beta_1^\dagger \mathbf{a}(\omega_2)|^2 = \frac{1 + Si(M\omega_d)}{2} \approx 1 - \frac{1}{4}\Delta^2 + \frac{3}{80}\Delta^4 \quad (B.24)$$

$$|\tilde{\beta}_2^\dagger \mathbf{a}(\omega_1)|^2 = |\beta_2^\dagger \mathbf{a}(\omega_2)|^2 = \frac{1 - Si(M\omega_d)}{2} \approx \frac{1}{4}\Delta^2 - \frac{3}{80}\Delta^4. \quad (B.25)$$

Similarly, for the midarrival angle  $\omega_m = (\omega_1 + \omega_2)/2$  we also have

$$|\tilde{\beta}_1^\dagger \mathbf{a}(\omega_m)|^2 = |\beta_1^\dagger \mathbf{a}(\omega_m)|^2 = \frac{2(Si(M\omega_d/2))^2}{1 + Si(M\omega_d)} \approx 1 - \frac{1}{80}\Delta^4 \quad (B.26)$$

and

$$|\tilde{\beta}_2^\dagger \mathbf{a}(\omega_m)|^2 = |\beta_2^\dagger \mathbf{a}(\omega_m)|^2 \approx 0. \quad (B.27)$$

Finally, with  $\xi = \tilde{\xi} \triangleq PM/\sigma^2 = M/\sigma^2$ , and using (B.21) - (B.27) in (29) - (30), for bias we have

$$\eta(\omega_i) = \frac{M-2}{N} \left[ \frac{1}{\xi} + \frac{\Delta^{-2}}{\xi^2} \right] \quad (\text{B.28})$$

$$\tilde{\eta}(\omega_i) = \frac{M-2}{N} \left[ \frac{1}{\bar{\xi}} \left( \frac{M\Delta^{-2}}{6(M-2)} + \frac{1}{10} \right) + \frac{\Delta^{-2}}{\bar{\xi}^2} \left( \frac{2M\Delta^{-4}}{9(M-4)} - \frac{M\Delta^{-2}}{30(M-4)} \right) \right] \quad (\text{B.29})$$

for  $i = 1, 2$ , and for the midarrival angle  $\omega_m$ , this is given as

$$E[\hat{Q}(\omega_m)] = a + \frac{1}{N} \left[ \frac{1}{\xi} b + \frac{1}{\xi^2} c \right] \quad (\text{B.30})$$

in an uncorrelated source scene and

$$E[\hat{Q}(\omega_m)] = \bar{a} + \frac{1}{N} \left[ \frac{1}{\bar{\xi}} \bar{b} + \frac{1}{\bar{\xi}^2} \bar{c} \right] \quad (\text{B.31})$$

in a coherent scene. Here

$$\begin{aligned} a &= \bar{a} = Q(\omega_m) \approx \Delta^4/80, \\ b &\approx \left[ \frac{M-2}{2} (1 + \Delta^2/4) \right], \quad c \approx \left[ \frac{M-2}{4} (1 + \Delta^2/2) \right], \\ \bar{b} &\approx \frac{M-2}{4} \left[ 1 + \left( 1 - \frac{3}{2M} \right) \Delta^2 \right] \text{ and } \bar{c} \approx \frac{M-2}{16} \left[ 1 + \left( 2 - \frac{3}{M} \right) \Delta^2 \right] - \frac{M\Delta^4}{45(M-4)}. \end{aligned}$$

As a consequence of the above analysis, we also have

$$\sigma^2(\omega_m) = \frac{d}{N} \left[ \frac{1}{2\xi} e + \frac{1}{4\xi^2} f \right] \quad (\text{B.32})$$

and

$$\bar{\sigma}^2(\omega_m) = \frac{\bar{d}}{N} \left[ \frac{1}{4\bar{\xi}} \bar{e} + \frac{1}{16\bar{\xi}^2} \bar{f} \right] \quad (\text{B.33})$$

where

$$d = \bar{d} \approx \frac{\Delta^4}{40} \left( 1 - \frac{\Delta^4}{80} \right)$$

$$e \approx 1 + \Delta^2/4 + \Delta^4/40, \quad f \approx 1 + \Delta^2/2 + 9\Delta^4/80$$

$$\bar{e} \approx 1 + (1 - 3/2M)\Delta^2 + (47/80 - 15/8M + 9/4M^2)\Delta^4$$



$$\bar{f} \approx 1 + (2 - 3/M) \Delta^2 + (87/40 - 27/4M + 27/4M^2) \Delta^4$$

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## **2.3 GEESE (GEneralized Eigenvalues utilizing Signal subspace**

### **Eigenvectors ) – New Performance Results on Direction Finding**

#### **2.3.1 Introduction**

In recent times, multiple signal identification using multiple sensor elements has been a topic of considerable research interest in array signal processing. A variety of high resolution techniques that evaluate the directions-of-arrival of incoming planar wavefronts by exploiting certain eigenstructure properties associated with the sensor array output covariance matrix have been developed in this context [1] – [11]. Of these, the relatively new scheme called ESPRIT (Estimation of Signal Parameters via Rotational Invariance Techniques) [9] – [10] departs from its predecessors on several important accounts. It utilizes an underlying rotational invariance among signal subspaces induced by subsets of an array of sensors. To accomplish this, in the original ESPRIT scheme [9], the interelement covariances among the given sensors are used to construct the auto- and cross-covariance matrices first and the common noise variance is then evaluated by an eigendecomposition of the auto-covariance matrix. After subtracting the noise variance from the proper elements of the auto- and cross-covariance matrices, the generalized eigenvalues of a matrix pencil formed from the subtracted matrices are computed and they in turn are shown to uniquely determine the unknown directions-of-arrival [9] – [10]. Compared to the Multiple Signal Classification (MUSIC) technique [2], the ESPRIT scheme is known to reduce the computation and storage costs dramatically. In addition, this method is also shown to be more robust with regard to array imperfections than most of the earlier ones.

Notwithstanding these merits, when estimates of the inter-element covariances are used in these computations, subtracting the estimated noise variance from the auto- and cross-covariance matrices can at times be critical and may result in overall

inferior results. To circumvent this difficulty, the TLS-ESPRIT (Total Least Squares ESPRIT) scheme makes use of certain overlapping subarray output- and their cross-covariance matrices simultaneously [11]. Though TLS-ESPRIT is superior in its performance compared to ESPRIT, it is computationally much more complex. However, computational simplicity can be maintained without sacrificing superior performance by exploiting the underlying rotational invariance among signal subspace in an efficient manner. In that respect, the GEESE (Generalized Eigenvalues utilizing Signal Subspace Eigenvectors) technique – that is studied here seems to be promising. Unfortunately, unlike the MUSIC scheme [12], no statistical performance analysis results are presently available for the ESPRIT or the TLS-ESPRIT scheme reported in [9] – [10] to evaluate their imperfections.

In what follows, we first derive this proposed scheme for estimating the directions-of-arrival of multiple sources. This is carried out by observing a well-known property of the signal subspace; i.e., the subspace spanned by the true direction vectors is identically the same as the one spanned by the eigenvectors corresponding to all, except the smallest set of repeating eigenvalue of the array output covariance matrix. This elementary observation forms the basis for the algorithm described in section 2.3.2. Using results derived in appendix A [16], section 2.3.3 presents a first-order perturbation analysis for the case where the covariances are estimated from the data and evaluates the mean and variance of the directions-of-arrival estimators for a single-source scene and a two-source scene. These results are in turn used in deriving resolution thresholds associated with two closely-spaced equipowered sources.

In an uncorrelated and identical sensor noise situation, when exact covariances are available, all these techniques can be applied to a uniformly placed array or a pairwise matched arbitrary array with codirectional sensor doublets. Since functionally these two arrays generate the same structured data with respect to the methods under discussion in the exact case, we will assume a uniform array to describe the algorithm.

### 2.3.2 Problem Formulation

Let an uniform array consisting of  $M$  sensors receive signals from  $K$  narrowband sources  $u_1(t), u_2(t), \dots, u_K(t)$ , which are at most partially correlated. Further, the respective arrival angles are assumed to be  $\theta_1, \theta_2, \dots, \theta_K$  with respect to the line of the array. Using complex signal representation, the received signal  $x_i(t)$  at the  $i^{th}$  sensor can be expressed as

$$x_i(t) = \sum_{k=1}^K u_k(t) e^{-j\pi(i-1)\cos\theta_k} + n_i(t). \quad (1)$$

Here the interelement distance is taken to be half the wavelength common to all signals and  $n_i(t)$  represents the additive noise at the  $i^{th}$  sensor. It is assumed that the signals and noises are stationary, zero-mean circular Gaussian independent random processes, and further the noises are assumed to be independent and identical between themselves with common variance  $\sigma^2$ . Rewriting (1) in common vector notation and with  $\omega_k = \pi \cos \theta_k$  ;  $k = 1, 2, \dots, K$ , we have

$$\mathbf{x}(t) = \sqrt{M} \sum_{k=1}^K u_k(t) \mathbf{a}(\omega_k) + \mathbf{n}(t), \quad (2)$$

where  $\mathbf{x}(t)$  is the  $M \times 1$  vector

$$\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_M(t)]^T, \quad (3)$$

and  $\mathbf{a}(\omega_k)$  is the normalized direction vector associated with the arrival angle  $\theta_k$ ; i.e.,

$$\mathbf{a}(\omega_k) = \frac{1}{\sqrt{M}} [1, \mu_k^{-1}, \mu_k^{-2}, \dots, \mu_k^{-(M-1)}]^T, \quad (4)$$

with

$$\mu_k = e^{j\omega_k}. \quad (5)$$

The array output vector  $\mathbf{x}(t)$  can further be rewritten as<sup>(1)</sup>

$$\mathbf{x}(t) = \mathbf{A} \mathbf{u}(t) + \mathbf{n}(t), \quad (6)$$

where

$$\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_K(t)]^T \quad (7)$$

$$\mathbf{n}(t) = [n_1(t), n_2(t), \dots, n_M(t)]^T \quad (8)$$

and

$$\mathbf{A} = \sqrt{M} [\mathbf{a}(\omega_1), \mathbf{a}(\omega_2), \dots, \mathbf{a}(\omega_K)]. \quad (9)$$

Here  $\mathbf{A}$  is an  $M \times K$  matrix with Vandermonde-structured columns ( $M > K$ ) of rank  $K$ .

From our assumptions it follows that the array output covariance matrix has the form

$$\mathbf{R} \triangleq E [\mathbf{x}(t) \mathbf{x}^\dagger(t)] = \mathbf{A} \mathbf{R}_u \mathbf{A}^\dagger + \sigma^2 \mathbf{I} \quad (10)$$

where

$$\mathbf{R}_u \triangleq E [\mathbf{u}(t) \mathbf{u}^\dagger(t)] \quad (11)$$

represents the source covariance matrix which remains as nonsingular so long as the sources are at most partially correlated. In that case  $\mathbf{A} \mathbf{R}_u \mathbf{A}^\dagger$  is also of rank  $K$  and hence, if  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M$  and  $\beta_1, \beta_2, \dots, \beta_M$  represent the eigenvalues and the corresponding eigenvectors of  $\mathbf{R}$  respectively, i.e.,

$$\mathbf{R} = \sum_{i=1}^M \lambda_i \beta_i \beta_i^\dagger, \quad (12)$$

then the above rank property implies that  $\lambda_{K+1} = \lambda_{K+2} = \dots = \lambda_M = \sigma^2$ . As a result for  $i = K+1, K+2, \dots, M$

$$\mathbf{R} \beta_i = (\mathbf{A} \mathbf{R}_u \mathbf{A}^\dagger + \sigma^2 \mathbf{I}) \beta_i = \sigma^2 \beta_i$$

(1) Here onwards  $^T, (\cdot)^T \triangleq \cdot^\dagger$  stand for the transpose and the complex conjugate transpose of  $\cdot$ , respectively.

or equivalently,

$$\mathbf{A}\mathbf{R}_u\mathbf{A}^\dagger\boldsymbol{\beta}_i = \mathbf{0}. \quad (13)$$

For full column rank  $\mathbf{A}$  matrix and nonsingular  $\mathbf{R}_u$ , (13) implies  $\mathbf{A}^\dagger\boldsymbol{\beta}_i = \mathbf{0}$ , or

$$\boldsymbol{\beta}_i^\dagger \mathbf{a}(\omega_k) = 0, \quad i = K+1, \dots, M, \quad k = 1, 2, \dots, K. \quad (14)$$

Schmidt [2] has used this well-known property in defining the estimator

$$P(\omega) = \frac{1}{Q(\omega)} \quad (15)$$

where

$$Q(\omega) = \sum_{i=K+1}^M |\boldsymbol{\beta}_i^\dagger \mathbf{a}(\omega)|^2. \quad (16)$$

Notice that  $Q(\omega) = 0$  iff  $\omega \in \{\omega_1, \omega_2, \dots, \omega_K\}$  and consequently the peaks of (15) correspond to the true directions of arrival.

A completely different point of view can be developed using (14). Since the  $K$  true direction vectors  $\mathbf{a}(\omega_1), \mathbf{a}(\omega_2), \dots, \mathbf{a}(\omega_K)$  are linearly independent, they span a  $K$  dimensional proper subspace called the signal subspace. Further, from (14), this subspace is orthogonal to the subspace spanned by the eigenvectors  $\boldsymbol{\beta}_{K+1}, \boldsymbol{\beta}_{K+2}, \dots, \boldsymbol{\beta}_M$ , implying that the signal subspace spanned by  $\mathbf{a}(\omega_1), \mathbf{a}(\omega_2), \dots, \mathbf{a}(\omega_K)$  coincides with that spanned by the eigenvectors  $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_K$ . Using this crucial observation, the eigenvectors  $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_K$  in the signal subspace can be expressed as a linear combination of the true direction vectors (columns of  $\mathbf{A}$ ); i.e.,

$$\boldsymbol{\beta}_l = \sum_{k=1}^K \tilde{c}_{kl} \mathbf{a}(\omega_k), \quad l = 1, 2, \dots, K. \quad (17)$$

Define the  $M \times K$  signal subspace eigenvector matrix as

$$\mathbf{B} \triangleq [\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_K]. \quad (18)$$

Using (17)

$$\mathbf{B} = \left[ \sum_{k=1}^K \tilde{c}_{k1} \mathbf{a}(\omega_k), \sum_{k=1}^K \tilde{c}_{k2} \mathbf{a}(\omega_k), \dots, \sum_{k=1}^K \tilde{c}_{kK} \mathbf{a}(\omega_k) \right] = \mathbf{A} \tilde{\mathbf{C}} \quad (19)$$

where  $\mathbf{A}$  is as defined in (9) and  $\tilde{\mathbf{C}}$  is a  $K \times K$  nonsingular matrix whose  $(i, j)^{th}$  element is  $\tilde{c}_{ij} / \sqrt{M}$ . Further, define two matrices  $\mathbf{B}_1$  and  $\mathbf{B}_2$  using the first  $L$  rows and the  $2^{nd}$  to  $(L+1)^{th}$  rows of  $\mathbf{B}$  respectively where  $K \leq L \leq M-1$ ; i.e.,

$$\mathbf{B}_1 = \left[ \mathbf{I}_L \mid \mathbf{O}_{L, M-L} \right] \mathbf{B} \quad (20)$$

and

$$\mathbf{B}_2 = \left[ \mathbf{O}_{L,1} \mid \mathbf{I}_L \mid \mathbf{O}_{L, M-L-1} \right] \mathbf{B}. \quad (21)$$

Then, we have the following interesting result<sup>(2)</sup>

*Theorem: Let  $\gamma_i$  represent the generalized singular values associated with the matrix pencil  $\{\mathbf{B}_1, \mathbf{B}_2\}$ . Then*

$$\gamma_k = \mu_k, \quad k = 1, 2, \dots, K. \quad (22)$$

*Proof:* From (20) and (21),

$$\mathbf{B}_1 = \mathbf{A}_1 \tilde{\mathbf{C}}, \quad \mathbf{B}_2 = \mathbf{A}_2 \tilde{\mathbf{C}} \quad (23)$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \mu_1^{-1} & \mu_2^{-1} & \dots & \mu_K^{-1} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \mu_1^{-(L-1)} & \mu_2^{-(L-1)} & \dots & \mu_K^{-(L-1)} \end{bmatrix} \quad (24)$$

(2) Here  $\mathbf{I}_K$  represents the  $K \times K$  identity matrix and  $\mathbf{O}_{K,J}$  represents the  $K \times J$  matrix with all zero entries.



and

$$\mathbf{A}_2 = \begin{bmatrix} \mu_1^{-1} & \mu_2^{-1} & \cdots & \mu_K^{-1} \\ \mu_1^{-2} & \mu_2^{-2} & \cdots & \mu_K^{-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1^{-L} & \mu_2^{-L} & \cdots & \mu_K^{-L} \end{bmatrix} = \mathbf{A}_1 \mathbf{D} \quad (25)$$

with

$$\mathbf{D} = \text{diag}[\mu_1^{-1}, \mu_2^{-1}, \dots, \mu_K^{-1}]. \quad (26)$$

Notice that  $\mathbf{A}_1, \mathbf{A}_2$  are matrices of size  $L \times K$  and  $\mathbf{D}$  is of size  $K \times K$ . To obtain the generalized singular values for the matrix pencil  $\{\mathbf{B}_1, \mathbf{B}_2\}$ , using the above representation we have

$$\mathbf{B}_1 - \gamma \mathbf{B}_2 = \mathbf{A}_1 \tilde{\mathbf{C}} - \gamma \mathbf{A}_1 \mathbf{D} \tilde{\mathbf{C}} = \mathbf{A}_1 (\mathbf{I}_K - \gamma \mathbf{D}) \tilde{\mathbf{C}} \quad (27)$$

Since the  $K$  columns of  $\mathbf{B}$  in (19) are independent,  $\mathbf{B}$  is of rank  $K$  ( $M > K$ ). Moreover, from the definitions of the rectangular matrices  $\mathbf{B}_1, \mathbf{B}_2$  in (20), (21), these matrices are also of rank  $K$  (full column rank) and using (23),  $\text{rank}(\mathbf{A}_1) = \text{rank}(\tilde{\mathbf{C}}) = K$  since  $L \geq K$ . Thus, from (27), the singular values of the above matrix pencil  $\{\mathbf{B}_1, \mathbf{B}_2\}$  are given by the roots of

$$|\mathbf{I}_K - \gamma \mathbf{D}| = 0. \quad (28)$$

These generalized singular values correspond to the complex conjugates of the diagonal elements of  $\mathbf{D}$ ; i.e.,

$$\gamma_k = \mu_k, \quad k = 1, 2, \dots, K. \quad (29)$$

Notice that  $L$  can be any integer between  $K$  and  $M - 1$ .

Q.E.D

So far we have proceeded under the assumption that the ensemble average of the array output covariance matrix is available. It may be remarked that the underlying rotational invariance idea that has been exploited here, is basic to the ESPRIT scheme and in that sense they are equivalent when covariances are exactly known. Usually, these exact averages are unknown and in practice, the estimates obtained from the array output data are used. In that case these methods give rise to different algorithms, and in actual practice, these algorithms will perform differently. Often the maximum likelihood (ML) procedure is employed in computing these covariance estimates. For zero-mean  $M$ -variate circular Gaussian data  $\mathbf{x}(t_n)$ ,  $n = 1, 2, \dots, N$  in (6), with unknown  $M \times M$  covariance matrix  $\mathbf{R}$ , the ML estimate  $\mathbf{S}$  of the covariance matrix is given by [15]

$$\mathbf{S} = \frac{1}{N} \sum_{n=1}^N \mathbf{x}(t_n) \mathbf{x}^\dagger(t_n). \quad (30)$$

The eigendecomposition of  $\mathbf{S}$  given by

$$\mathbf{S} = \mathbf{E} \mathbf{L} \mathbf{E}^\dagger, \quad \mathbf{E} \mathbf{E}^\dagger = \mathbf{I} \quad (31)$$

where

$$\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_K, \dots, \mathbf{e}_M]; \quad \mathbf{L} = \text{diag}[l_1, l_2, \dots, l_M].$$

is usually used to obtain the  $L \times K$  matrices  $\mathbf{E}_1, \mathbf{E}_2$  by replacing  $\mathbf{B}$  by  $\mathbf{E}$  in (20) and (21). The estimates  $\hat{\gamma}_i, i = 1, 2, \dots, K$  of the true angular parameters  $\gamma_i, i = 1, 2, \dots, K$  are then given by the generalized eigenvalues of the matrix pencil  $\{\mathbf{E}_1, \mathbf{E}_2\}$ . Simulation results using this procedure is presented in Fig.1 for a three-source scene with details as indicated there.

In what follows, we establish several performance results of the algorithm presented here using the statistical properties of the estimated covariance matrix  $\mathbf{S}$  derived in [16, 17]. These results are in turn used in analyzing the performance of the estimated generalized eigenvalues in a single-source and a two-source scene.

### 2.3.3 Performance Analysis

In this section, we examine the statistical behavior of the estimated generalized eigenvalues in a single-source case and a two-source case for the least favorable configuration  $L = K$  as well as the most favorable configuration  $L = M - 1$ . These results are subsequently used in deriving associated threshold expressions for resolving two closely-spaced sources. For  $L = K$ , it is shown here that the bias of the estimated generalized eigenvalues is zero and the variance is nonzero within a  $1/N$  approximation. This behavior is unlike the MUSIC scheme where within a  $1/N$  approximation, the bias is nonzero and the variance is zero [12, 16]. For  $L > K$ , the situation is considerably more complicated. In particular, it is also shown here that for  $L = M - 1$  the estimated generalized eigenvalues are no longer unbiased in a two-source scene. For sake of completeness, the exact bias expressions together with their variances are also given. We begin by considering the  $L = K$  case.

#### Case 1 : The Least Favorable Configuration ( $L = K$ )

##### A. Single-Source Scene

In Appendix A, the mean and variance of the estimated generalized eigenvalue  $\hat{\gamma}_1$  for a single-source scene are shown to be

$$E [\hat{\gamma}_1] = \gamma_1 + o\left(\frac{1}{N\sqrt{N}}\right) \quad (32)$$

and

$$Var(\hat{\gamma}_1) = \frac{2M \lambda_1 \sigma^2}{N (\lambda_1 - \sigma^2)^2} + o\left(\frac{1}{N\sqrt{N}}\right). \quad (33)$$

Here,  $o(1/N\sqrt{N})$  represents the terms of order less than  $1/N$ . From these results,  $\hat{\gamma}_1$  is unbiased within a first-order approximation.

With  $\lambda_1 = M P + \sigma^2$  for the signal subspace eigenvalue where  $P$  represents the signal power, (33) simplifies into

$$Var(\hat{\gamma}_1) = \frac{2M}{N} \left[ \frac{1}{\xi} + \frac{1}{\xi^2} \right] + o\left(\frac{1}{N\sqrt{N}}\right) \quad (34)$$

where  $\xi = M P / \sigma^2$  represents the array output signal-to-noise ratio.

## B. Two-Source Scene

With the help of (A.1), the mean and variance of the estimated generalized eigenvalues  $\hat{\gamma}_i, i = 1, 2$  in two equipowered uncorrelated source scene are shown to be (see Appendix B)

$$E[\hat{\gamma}_i] = \gamma_i + o\left(\frac{1}{N\sqrt{N}}\right) ; i = 1, 2, \quad (35)$$

and

$$Var(\hat{\gamma}_i) = \frac{M \left[ 2 + \text{Re}(\mu_1^{-1} \mu_2) \right]}{2N \left[ 1 - \text{Re}(\mu_1^{-1} \mu_2) \right]} \left[ (1 + |\rho_s|) \frac{\lambda_1 \sigma^2}{(\lambda_1 - \sigma^2)^2} + (1 - |\rho_s|) \frac{\lambda_2 \sigma^2}{(\lambda_2 - \sigma^2)^2} \right] + o\left(\frac{1}{N\sqrt{N}}\right) ; i = 1, 2 \quad (36)$$

where

$$\rho_s = \mathbf{a}^\dagger(\omega_1) \mathbf{a}(\omega_2) = e^{j(M-1)\omega_d} \frac{\sin M \omega_d}{M \sin \omega_d}, \quad \omega_d = (\omega_1 - \omega_2)/2. \quad (37)$$

Thus, for  $L = K$ , within a  $1/N$  approximation,  $\hat{\gamma}_i, i = 1, 2$  are once again unbiased estimates with finite variance. Simulation results presented in Fig.2 are seen to be in agreement with these conclusions. The random pattern for actual bias in Fig.2 may be attributed to computational and other round-off errors and indicates the absence of  $1/N$  term there.

To simplify (36) further, for two equipowered uncorrelated sources, the signal subspace eigenvalues are given by [12]

$$\lambda_i = M P (1 \pm |\rho_s|) + \sigma^2, \quad i = 1, 2. \quad (38)$$

With (38) in (36), finally it simplifies into

$$Var(\hat{\gamma}_i) = \frac{M(2 + \cos 2\omega_d)}{N(1 - \cos 2\omega_d)} \left[ \frac{1}{\xi} + \frac{1}{\xi^2(1 - |\rho_s|^2)} \right] + o\left(\frac{1}{N\sqrt{N}}\right) \quad i = 1, 2. \quad (39)$$

These expressions can be used to determine the resolution threshold associated with two closely spaced sources. For a specific input SNR, the resolution threshold represents the minimum amount of angular separation required to identify the sources as separate entities unambiguously. From (35) and (39), since the standard derivation of  $\hat{\gamma}_i, i = 1, 2$  is substantially larger than their respective bias, it is clear that the resolution threshold is mostly determined by the behavior of the standard deviation. In order to obtain a measure of the resolution threshold for two closely spaced sources, consider the situation shown in Fig.3. Evidently, the sources are resolvable if  $\hat{\gamma}_1$  and  $\hat{\gamma}_2$  are both inside the cones  $C_1$  and  $C_2$  respectively or equivalently if  $|\arg(\hat{\gamma}_i) - \arg(\gamma_i)| < \omega_d, i = 1, 2$ . Exact calculations based on this criteria turns out to be rather tedious. But as computation results in Fig.1 show,  $\hat{\gamma}_1$  and  $\hat{\gamma}_2$  are usually within a small circular neighborhood centered about  $\gamma_1$  and  $\gamma_2$ . This suggests a more conservative criterion for resolution; i.e., the sources are resolvable if  $\hat{\gamma}_1$  and  $\hat{\gamma}_2$  are both inside the circles  $c_1$  and  $c_2$  respectively in Fig.3. In that case, the maximum value of the common radii of these circles is easily shown to be  $\sin \omega_d$ . Thus, at an SNR satisfying

$$\sqrt{Var(\hat{\gamma}_i)} = l \sin \omega_d \quad (40)$$

where  $l$  is some positive integer, using (40) we finally have the associated threshold SNR to be

$$\xi_{l,K} = \frac{M(2 + \cos 2\omega_d)}{2l^2N(1 - \cos 2\omega_d)\sin^2\omega_d} \left[ 1 + \left( 1 + \frac{4l^2N(1 - \cos 2\omega_d)\sin^2\omega_d}{M(2 + \cos 2\omega_d)(1 - |\rho_s|^2)} \right)^{1/2} \right]. \quad (41)$$

This threshold SNR can also be expressed in terms of the "effective angular separation" parameter  $\Delta^2$  given by [12]

$$\Delta^2 = M^2 \omega_d^2 / 3.$$

For closely-spaced sources,  $M^2 \omega_d^2 \ll 1$  and in that case since

$$\cos 2\omega_d \approx 1 - \frac{6\Delta^2}{M^2} + \frac{6\Delta^4}{M^4} \quad (42)$$

and

$$|\rho_s|^2 \approx 1 - \Delta^2 + \frac{2}{5}\Delta^4, \quad (43)$$

using (42) and (43) in (41), we have

$$\xi_{l,K} \approx \frac{M}{6l^2N} \left[ \frac{M^4}{2\Delta^4} - \frac{M^2}{\Delta^2} \right] \left[ 1 + \left( 1 + \frac{24l^2N\Delta^2}{M^5} \right)^{1/2} \right] \quad (44)$$

Notice that calculations for  $\text{Var}(\hat{\gamma}_i)$  in (39) has been carried out for  $L = K (=2)$  case and hence the above threshold expression also corresponds to this least favorable configuration, which only uses part of the available signal subspace eigenvector information in its computations. When higher value of  $L$  is used to evaluate  $\hat{\gamma}_i$ , the corresponding threshold expressions also should turn out to be superior to that in (41). These conclusions are seen to closely agree with results of simulation presented in Table 1. Similar threshold comparisons are carried out in Fig.4 for the MUSIC scheme and the GEESE scheme. In the case of two uncorrelated sources, by equating the actual bias at the true arrival and middle angles Kaveh *et al.* has shown the resolution threshold for the standard MUSIC scheme to be [12]

$$\xi_M \approx \frac{1}{N} \left[ \frac{20(M-2)}{\Delta^4} \left( 1 + \left[ 1 + \frac{N}{5(M-2)} \Delta^2 \right]^{1/2} \right) \right]. \quad (45)$$

Fig.4 shows such a comparison using (44) with  $l = 2$  and (45). From Table 1, the

corresponding SNR values are observed to have at least 30 percent probability of resolution.

### Case 2 : The Most Favorable Configuration ( $L = M - 1$ )

For  $L > K$ , the situation is much more complex and the estimator,  $\hat{\gamma}_i, i = 1, 2$  are no longer unbiased within  $1/N$  approximation. The exact bias and variance expressions can be computed by proceeding as in Appendix B. These computations have been carried out for the most favorable configuration ( $L = M - 1$ ) in a single-source scene as well as a two-source scene and the results are summarized below [18].

#### A. Single-Source Scene

The mean and variance of the generalized eigenvalue  $\hat{\gamma}_1$  in this case with  $E[|u_1|^2] = 1$  are

$$E[\hat{\gamma}_1] = \gamma_1 + \frac{(M + \sigma^2)\sigma^2}{N} \frac{M - 2}{(M - 1)^2} \gamma_1 + o(1/N^2) \quad (46)$$

and

$$Var(\hat{\gamma}_1) = \frac{2[2(M - 1)(1 - \mu_1 \cos \omega_1) + M \mu_1 \cos \omega_1]}{N(M - 1)^2} \left[ \frac{1}{\xi} + \frac{1}{\xi^2} \right] + o(1/N^2). \quad (47)$$

Interestingly enough, the above estimate is biased even within a  $1/N$  approximation. However the bias and variance values in (46) - (47) are quite small compared to the variance (34) in a similar situation for the least favorable configuration.

#### B. Two-Source Scene

Once again, starting with (A.1), after a long series of algebraic manipulations, the mean and variance of the generalized eigenvalues  $\hat{\gamma}_i, i = 1, 2$ , in two equipowered uncorrelated source scene can be shown to be

$$E[\hat{\gamma}_i] = j \frac{Me^{j\omega_m}}{(M - 1)^6 \omega_d^3} \left[ \frac{1}{\xi} \tau_1(M, \omega_d) + \frac{1}{\xi^2(1 - |\rho_s|^2)} \tau_2(M, \omega_d) \right] \quad (48)$$

where

$$r_i(M, \omega_d) = p_i e^{-j2\omega_d} + q_i e^{-j\omega_d} + r_i \quad ; \quad i = 1, 2.$$

and the constants  $p_i, q_i, r_i; i = 1, 2$ , are as given in Appendix C.

$$Var(\hat{\gamma}_i) = \frac{\nu(M, \Delta^2)}{N} \left[ \frac{\Delta^2}{\xi} + \frac{1}{\xi^2} \right], \quad i = 1, 2 \quad (49)$$

where

$$\nu(M, \Delta^2) = \frac{2M^5}{(M-1)^6} \left[ \left( 3 - \frac{2}{M^2} \right) \frac{1}{\Delta^4} + \frac{1}{5} \left( 2 - \frac{14}{M} + \frac{123}{M^2} \right) \frac{1}{\Delta^2} - \left( 2 - \frac{7}{M} - \frac{15}{M^2} + \frac{95}{M^3} \right) \right]$$

The  $1/N$  dependence for mean and variance is clearly evident in the simulation results presented in Fig.6. The associated threshold SNR in this case can be obtained with the help of (40) and (49) and this gives

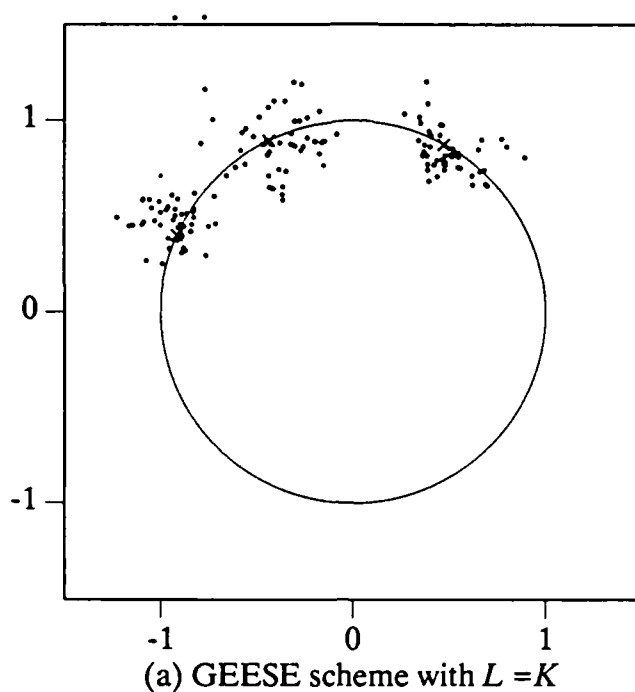
$$\xi_{l, M-1} \approx \frac{M^2 \nu(M, \Delta^2)}{6l^2 N} \left[ 1 + \left( 1 + \frac{2l^2 N (M-1)^6 \Delta^2}{M^5 (M^2 - 2)} \right)^{1/2} \right] \quad (50)$$

For the same source scene and probability of resolution discussed in Fig.4, new simulation results are presented there for this most favorable configuration. As remarked earlier, the SNR required to resolve two sources in this case is seen to be substantially smaller than that in the former case ( $L = K$ ). In particular, to resolve two closely spaced sources under identical conditions, in terms of input SNR, the most favorable configuration seems to require about 12dB less compared to the MUSIC scheme and about 18dB less compared to the least favorable configuration. Once again, utilization of all available information in this ( $L = M - 1$ ) case may be attributed to its superior performance. Fig.5 shows a new set of comparisons for another array length. From these results, it may be reasonably concluded that when all available signal subspace information is exploited, the proposed algorithm outperforms the MUSIC scheme.

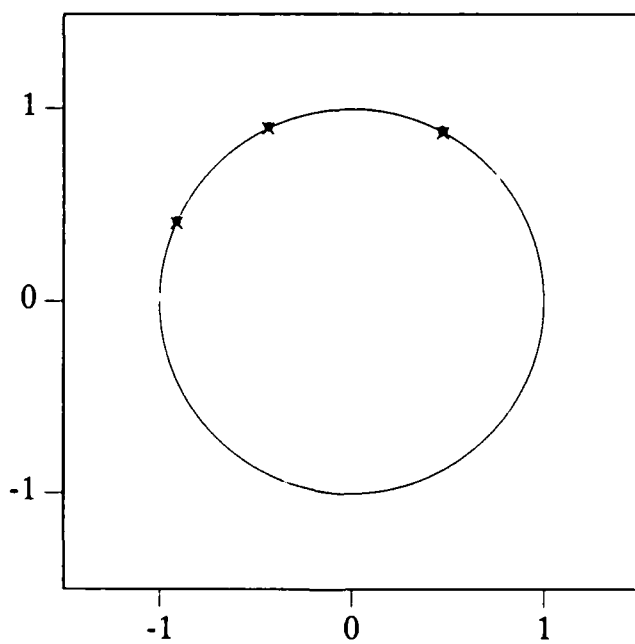


#### IV. Conclusions

This report analyzes a technique for estimating the directions-of-arrival of correlated signals by making use of certain matrices associated with the signal subspace eigenvectors of the array output covariance matrix. This is based on the well-known property that in the case of uncorrelated and identical noise field, the subspace spanned by the true direction vectors is identical to the signal subspace (i.e., the one spanned by the eigenvectors associated with all, except the lowest repeating eigenvalue of the array output covariance matrix). Using a first-order asymptotic analysis, it is shown here that the angle-of-arrival estimator in its least favorable form is unbiased and has nonzero variance in a two-source scene. Although the estimator in its most favorable configuration turns out to be biased, the associated resolution threshold in an equipowered two-source scene is shown to be substantially smaller than that corresponding to the standard MUSIC scheme. Similar comparisons can be performed in a coherent scene by first employing the standard forward/backward smoothing technique to decorrelate the signals followed by the method described here, to estimate their actual arrival angles. Once again, performance comparisons can be made for a two coherent source scene after working out the asymptotic results in a similar manner.



(a) GEESE scheme with  $L = K$



(b) GEESE scheme with  $L = M - 1$

Fig. 1. Simulation results for a mixed-source scene. Three sources are located at  $30^\circ$ ,  $50^\circ$  and  $70^\circ$ . The first two sources are uncorrelated and third source is correlated with the first and second sources with correlation coefficients  $0.5 + j0.29$  and  $0.21 + j0.43$ , respectively. A ten-element array is used to receive the signals. Input SNR is taken to be 10dB. (number of simulations = 50, number of samples = 100)

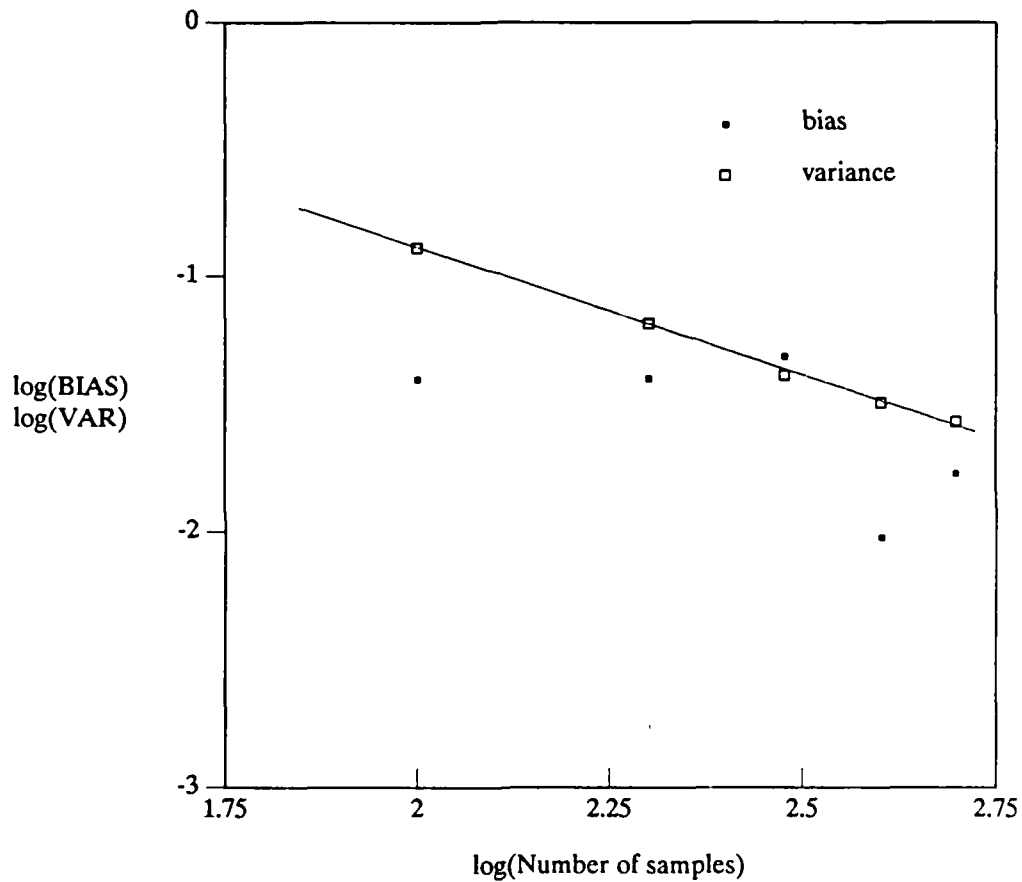


Fig. 2. Bias and Variance for the least favorable configuration ( $L = K (=2)$ )  
Bias and variance vs number of snapshots for two equipowered sources. A ten element array is used to receive signals from two sources located along  $45^\circ, 50^\circ$  with common SNR = 5dB. Here  $L = K (=2)$  and each simulation consists of 100 independent trials.

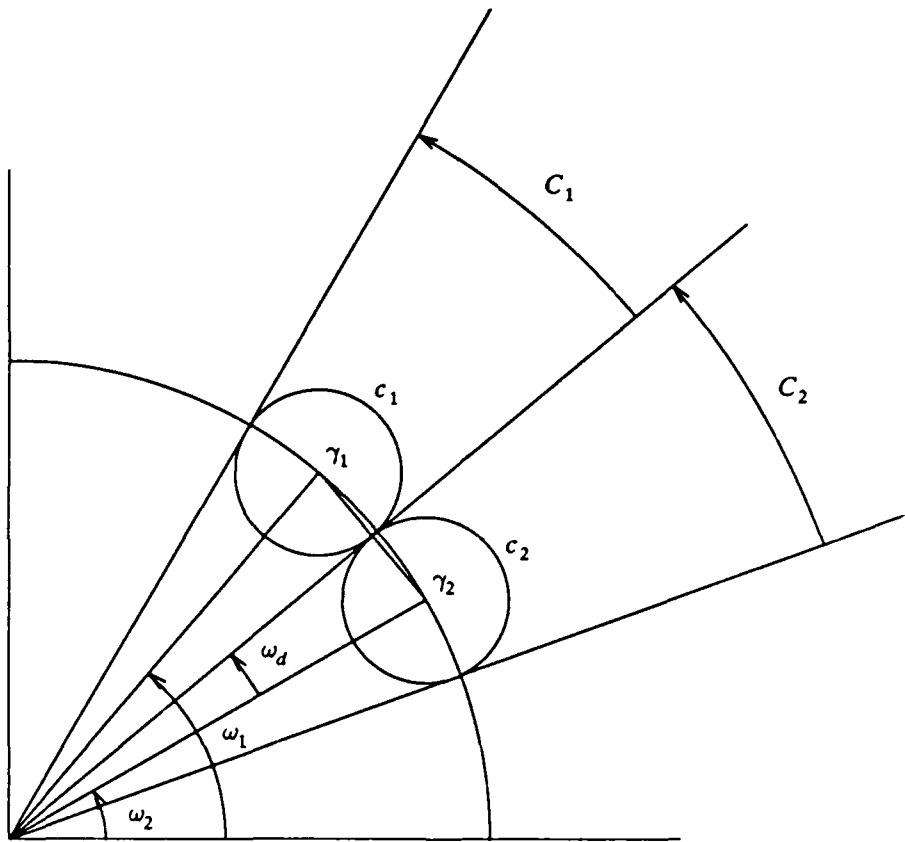


Fig. 3 Resolution Threshold Analysis

Table 1

Resolution threshold and probability of resolution vs. angular separation for two equipowered sources ( $K = 2$ ) in an uncorrelated scene. (number of sensors  $M = 7$ , number of snapshots = 100, number of simulations = 100)

angles of arrival		angular separation	SNR	Prob. of Resolution			
$\theta_1$ $\theta_2$		$2\omega_d$		MUSIC	GEESE		
					$L = K$	$L = 4$	$L = M - 1$
25.00	30.00	0.1265	4				0.14
			5				0.21
			6				0.33
			8				0.54
			16		0.12	0.64	0.98
			18	0.19	0.19	0.77	1.00
			19	0.31	0.20	0.84	
			20	0.42	0.20	0.89	
			21	0.63	0.20	0.96	
			22	0.71	0.25	0.98	
			23	0.80	0.31		
			24	0.91	0.40		
40.00	45.00	0.1852	0				0.27
			1				0.31
			2				0.43
			3				0.57
			9		0.09	0.58	0.99
			11	0.12	0.16	0.77	1.00
			12	0.30	0.17	0.83	
			13	0.32	0.21	0.91	
			14	0.48	0.23	0.94	
			15	0.73	0.26	0.97	
			16	0.75	0.30	0.99	
			17	0.84	0.31	1.00	
75.00	80.00	0.2676	-5				0.19
			-4				0.23
			-3				0.35
			-2				0.51
			-1			0.22	0.57
			0			0.28	0.66
			4		0.08	0.66	0.98
			6	0.29	0.11	0.85	1.00
			7	0.41	0.12	0.87	
			8	0.57	0.16	0.92	
			9	0.74	0.22	0.94	
			10	0.81	0.25	0.97	
			11	0.92	0.30	1.00	
			12	0.99	0.56		

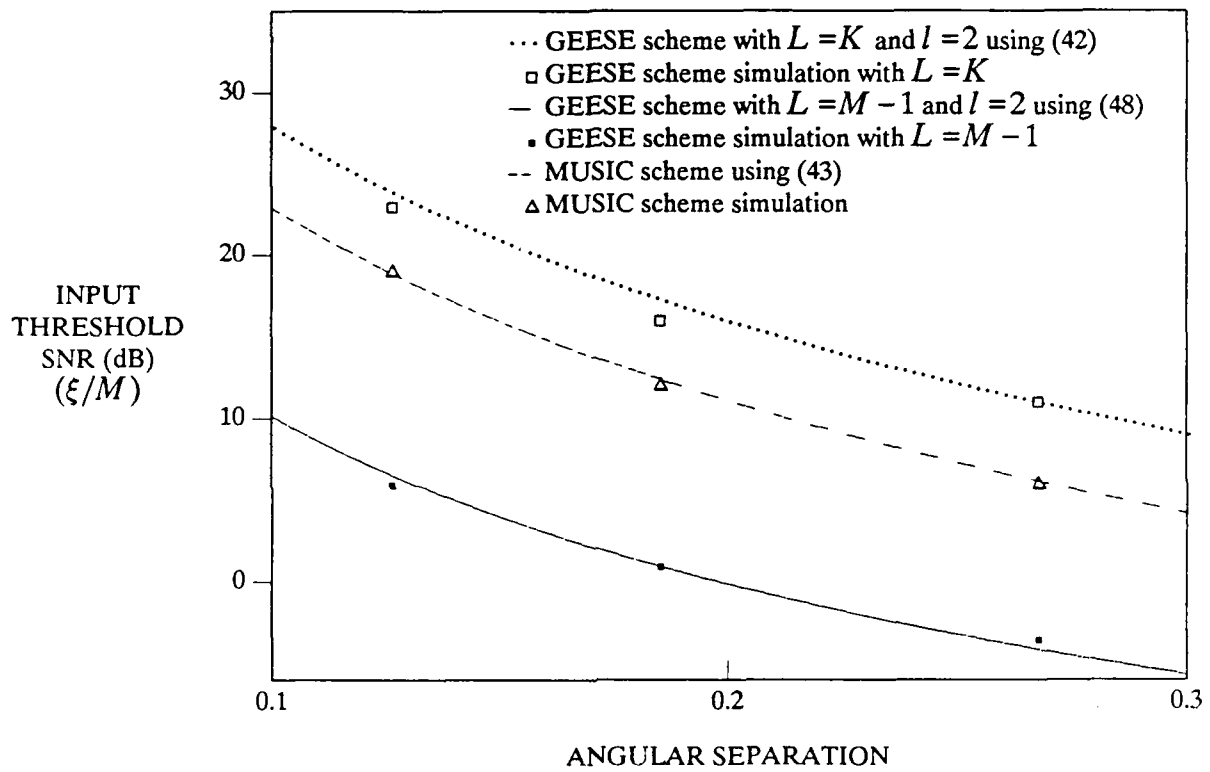


Fig. 4 Resolution threshold vs angular separation for two equipowered sources. A seven element array is used to receive signals in both cases. One hundred snapshots are taken for each simulation. In each simulation, the associated probability of resolution is 30 percent.

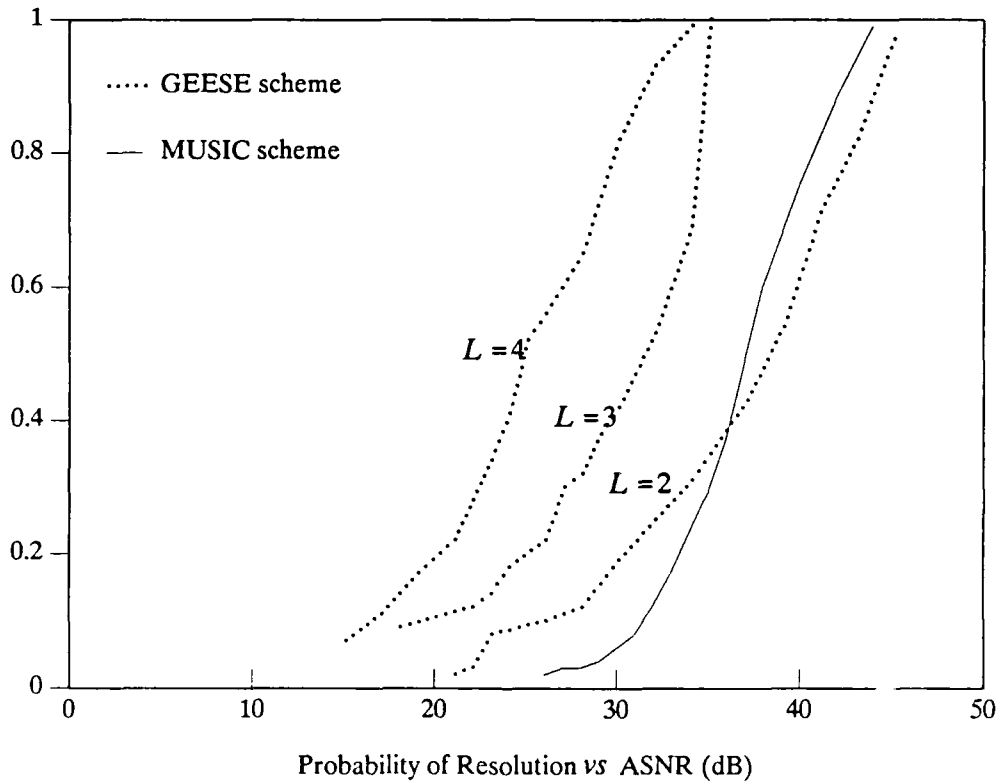


Fig. 5 Probability of resolution as a function of ASNR for two equipowered sources. Angular separation  $(\omega_1 - \omega_2)$  is taken to be 0.1 rad. A five-element array receives signals and one hundred simulations with 100 snapshots are used to obtain the probability of resolution for each ASNR. In each simulation, two sources are considered resolved if simultaneously  $|\arg(\hat{\gamma}_i) - \arg(\gamma_i)| < \omega_d$ ,  $i = 1, 2$  for the proposed GEESE scheme and  $\hat{Q}(\omega_i) < \hat{Q}(\omega_m)$ ,  $i = 1, 2$  for the MUSIC scheme.

Here  $\hat{Q}(\omega) = \sum_{i=K+1}^M |\mathbf{e}_i^\dagger \mathbf{a}(\omega)|^2$ .

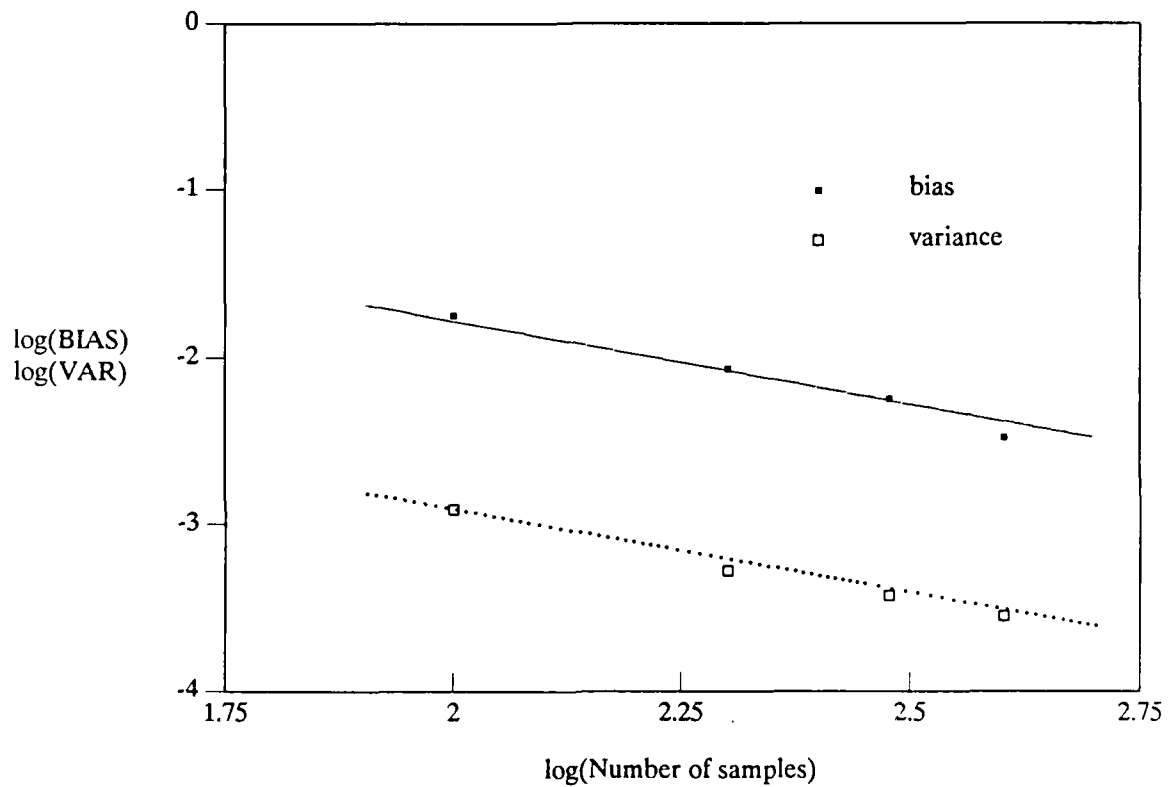


Fig.6 Bias and variance for the most favorable configuration ( $L = M - 1$ ) Bias and variance vs number of snapshots for two equipowered sources. A ten element array is used to receive signals along  $60^\circ$ ,  $65^\circ$ . In each simulation, the SNR taken to be 3dB. Here  $L = M - 1$  and each simulation consists of 500 independent trials.



## Appendix A

### Single-Source Scene Analysis

With  $\hat{e}_i$ ,  $i = 1, 2, \dots, K$  representing a certain set of normalized estimated eigenvectors associated with the signal subspace eigenvectors  $\beta_1, \beta_2, \dots, \beta_K$  from [12, 16, 17], we have

$$\hat{e}_i = k_i [\beta_i + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^M w_{ji} \beta_j] + o(1/N^2). \quad (A.1)$$

where

$$k_i = 1 - \frac{1}{2N} \sum_{\substack{j=1 \\ j \neq i}}^M |w_{ji}|^2 \quad (A.2)$$

Here  $w_{ij}$ ,  $i = 1, 2, \dots, M$ ,  $j = 1, 2, \dots, K$  are zero mean, asymptotically Gaussian random variables with

$$E[w_{ij} w_{kl}^*] = \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \delta_{ik} \delta_{jl} \quad ; \quad i \neq j \text{ and } k \neq l \quad (A.3)$$

and

$$E[w_{ij} w_{kl}] = \frac{\lambda_i \lambda_j}{(\lambda_j - \lambda_i)^2} \delta_{il} \delta_{jk} \quad ; \quad i \neq j \text{ and } k \neq l \quad (A.4)$$

In a single-source scene, the eigenvector corresponding to the largest eigenvalue of the array output covariance matrix is

$$\beta_1 = [\beta_{11}, \beta_{21}, \dots, \beta_{M1}]^T = \frac{1}{\sqrt{M}} [1, \mu_1^{-1}, \dots, \mu_1^{-(M-1)}]^T \quad (A.5)$$

and from (A.1) the estimator for  $\beta_1$  is given by

$$\hat{e}_1 = k_1 [\beta_1 + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \beta_j] + o(1/N^2). \quad (A.6)$$

This gives  $\hat{\gamma}_1$  to be

$$\hat{\gamma}_1 = \hat{c}_{11}/\hat{c}_{21} \approx (\beta_{11} + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \beta_{1j}) / \left( \beta_{21} (1 + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \beta_{2j} / \beta_{21}) \right) \quad (A.7)$$

For  $|\frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \beta_{2j} / \beta_{21}| \ll 1$ , we can simplify (A.7) as

$$\begin{aligned} \hat{\gamma}_1 &\approx \frac{1}{\beta_{21}} \left( \beta_{11} + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \beta_{1j} \right) \left( 1 - \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \beta_{2j} / \beta_{21} + \frac{1}{N} \sum_{\substack{i=1 \\ i \neq 1}}^M \sum_{\substack{j=1 \\ j \neq 1}}^M w_{i1} w_{j1} \beta_{2i} \beta_{2j} / \beta_{21}^2 \right) \\ &= \gamma_1 + \frac{1}{\sqrt{N}} \Gamma_{11} + \frac{1}{N} \Gamma_{21} + o(1/N^2) \end{aligned} \quad (A.8)$$

where

$$\Gamma_{11} = \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \left[ \beta_{1j} / \beta_{21} - \beta_{11} \beta_{2j} / \beta_{21}^2 \right] \quad (A.9)$$

$$\Gamma_{21} = \sum_{\substack{i=1 \\ i \neq 1}}^M \sum_{\substack{j=1 \\ j \neq 1}}^M w_{i1} w_{j1} \left[ \beta_{11} \beta_{2i} \beta_{2j} / \beta_{21}^3 - \beta_{1i} \beta_{2j} / \beta_{21}^2 \right]. \quad (A.10)$$

Since the limiting joint distribution of  $w_{ij}$ ,  $i = 1, 2, \dots, M$ ,  $j = 1, 2, \dots, K$  tend to be normal with zero-mean and the odd-order moments of zero-mean Gaussian random variable are zeros, the expected value of  $\Gamma_{11}$  is zero. Thus,

$$\begin{aligned} E[\hat{\gamma}_1] &= \gamma_1 + \frac{1}{N} E[\Gamma_{21}] + o(1/N^2) \\ &= \gamma_1 + \frac{1}{N} \sum_{\substack{i=1 \\ i \neq 1}}^M \sum_{\substack{j=1 \\ j \neq 1}}^M E[w_{i1} w_{j1}] \left[ \beta_{11} \beta_{2i} \beta_{2j} / \beta_{21}^3 - \beta_{1i} \beta_{2j} / \beta_{21}^2 \right] + o(1/N^2). \end{aligned} \quad (A.11)$$

With (A.4) in (A.11)

$$E[\hat{\gamma}_1] = \gamma_1 + o(1/N^2), \quad (\text{A.12})$$

which is an unbiased estimator for  $\gamma_1$ .

Further, with the help of (A.8), (A.12), we get

$$\begin{aligned} \text{Var}(\hat{\gamma}_1) &= E\left[\left|\frac{1}{\sqrt{N}}\Gamma_{11} + \frac{1}{N}\Gamma_{21}\right|^2\right] \\ &= E\left[\frac{1}{N}\Gamma_{11}\Gamma_{11}^* + \frac{2}{N\sqrt{N}}\text{Re}(\Gamma_{11}\Gamma_{21}^*)\right] + o(1/N^2). \end{aligned} \quad (\text{A.13})$$

The term inside the expected operator can be written as

$$\begin{aligned} &\frac{1}{N}\Gamma_{11}\Gamma_{11}^* + \frac{2}{N\sqrt{N}}\text{Re}(\Gamma_{11}\Gamma_{21}^*) = \\ &\frac{1}{N}\sum_{\substack{i=1 \\ i \neq 1}}^M \sum_{\substack{j=1 \\ j \neq 1}}^M w_{i1}w_{j1}^* \left[\beta_{1i}/\beta_{21} - \beta_{11}\beta_{2i}/\beta_{21}^2\right] \left[\beta_{1j}/\beta_{21} - \beta_{11}\beta_{2j}/\beta_{21}^2\right]^* + \\ &\frac{2}{N\sqrt{N}}\text{Re}\left[\sum_{\substack{k=1 \\ k \neq 1}}^M \sum_{\substack{i=1 \\ i \neq 1}}^M \sum_{\substack{j=1 \\ j \neq 1}}^M w_{k1}w_{i1}^*w_{j1}^* \left[\frac{\beta_{1k}}{\beta_{21}} - \frac{\beta_{11}\beta_{2k}}{\beta_{21}^2}\right] \left[\frac{\beta_{11}\beta_{2i}\beta_{2j}}{\beta_{21}^3} - \frac{\beta_{1i}\beta_{2j}}{\beta_{21}^2}\right]^*\right] \end{aligned} \quad (\text{A.14})$$

Once again noticing that the odd-order moments of zero-mean Gaussian random variable are zero, we have

$$\text{Var}(\hat{\gamma}_1) = \frac{1}{N}\sum_{\substack{i=1 \\ i \neq 1}}^M \sum_{\substack{j=1 \\ j \neq 1}}^M E[w_{i1}w_{j1}^*] \left[\beta_{1i}/\beta_{21} - \beta_{11}\beta_{2i}/\beta_{21}^2\right] \left[\beta_{1j}/\beta_{21} - \beta_{11}\beta_{2j}/\beta_{21}^2\right]^* + o(1/N^2). \quad (\text{A.15})$$

Using (A.3) in (A.15) and with (A.5) we have

$$\text{Var}(\hat{\gamma}_1) = \frac{1}{N}\sum_{k=2}^M \frac{M\lambda_1\lambda_k}{(\lambda_1 - \lambda_k)^2} |\beta_{1k} - \beta_{2k}/\beta_{21}|^2$$

$$= \frac{M \lambda_1 \sigma^2}{N (\lambda_1 - \sigma^2)^2} \sum_{k=2}^M \left[ |\beta_{1k}|^2 + |\beta_{2k}|^2 - 2\text{Re}(\mu_1 \beta_{1k}^* \beta_{2k}) \right] + o(1/N^2). \quad (\text{A.16})$$

To simplify this further, notice that

$$\sum_{k=1}^M \beta_{ik} \beta_{jk}^* = \delta_{ij}, \quad i = 1, 2, \dots, M \text{ and } j = 1, 2, \dots, M. \quad (\text{A.17})$$

(A.17) together with (A.5) gives

$$\sum_{k=2}^M |\beta_{1k}|^2 = \sum_{k=2}^M |\beta_{2k}|^2 = 1 - \frac{1}{M} \quad (\text{A.18})$$

and

$$\sum_{k=2}^M \beta_{1k}^* \beta_{2k} = -\mu_1^{-1}/M. \quad (\text{A.19})$$

Finally, with (A.18)–(A.19) in (A.16) we have

$$\text{Var}(\hat{\gamma}_1) = \frac{2M \lambda_1 \sigma^2}{N (\lambda_1 - \sigma^2)^2} + o(1/N^2). \quad (\text{A.20})$$

## Appendix B

### Two-Source Scene Analysis

For two equipowered uncorrelated sources, the eigenvectors in the signal subspace of the array output covariance matrix are given by [12]

$$\beta_i = \frac{\mathbf{u}_1 \pm \mathbf{u}_2}{\sqrt{2(1 \pm |\rho_s|)}} \quad \text{for } Si(M\omega_d) > 0, i = 1, 2 \quad (\text{B.1})$$

where

$$\rho_s \triangleq \mathbf{a}^\dagger(\omega_1)\mathbf{a}(\omega_2) = e^{j(M-1)\omega_d} \frac{\sin M\omega_d}{M \sin \omega_d} \triangleq e^{j(M-1)\omega_d} Si(M\omega_d) \quad (\text{B.2})$$

and

$$\mathbf{u}_1 = \mu_1^{(M-1)/2} \mathbf{a}(\omega_1), \quad \mathbf{u}_2 = \mu_2^{(M-1)/2} \mathbf{a}(\omega_2) \quad (\text{B.3})$$

with  $\mathbf{a}(\omega)$  as defined in (4). Using these eigenvectors and with the help of (19)–(21) and  $L = K$ , we form the matrix pencil  $\{\mathbf{B}_1, \mathbf{B}_2\}$  and compute the generalized eigenvalues of this matrix pencil; i.e.,

$$\left| \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} - \gamma \begin{bmatrix} \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} \right| = \left| \begin{bmatrix} \beta_{11} - \gamma\beta_{21} & \beta_{12} - \gamma\beta_{22} \\ \beta_{21} - \gamma\beta_{31} & \beta_{22} - \gamma\beta_{32} \end{bmatrix} \right| = 0 \quad (\text{B.4})$$

where  $\beta_{ij}$  represents the  $i^{\text{th}}$  element of the  $j^{\text{th}}$  eigenvector. Then, (B.4) can be simplified as

$$\Delta_1 \gamma^2 - \Delta_2 \gamma + \Delta_3 = 0 \quad (\text{B.5})$$

where  $\Delta_1, \Delta_2, \Delta_3$  are the determinants defined as

$$\Delta_1 \triangleq \begin{vmatrix} \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{vmatrix}, \quad \Delta_2 \triangleq \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{31} & \beta_{32} \end{vmatrix}, \quad \Delta_3 \triangleq \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix}. \quad (\text{B.6})$$

From (B.6) and (B.1)–(B.3), we find the following useful identities

$$\Delta_3 = \beta_{11}\beta_{22} - \beta_{21}\beta_{12} = \frac{1}{M(1 - |\rho_s|^2)^{1/2}} \mu_1^{(M-1)/2} \mu_2^{(M-1)/2} (\mu_1^{-1} - \mu_2^{-1}) \quad (\text{B.7})$$

$$\Delta_2 = \beta_{11}\beta_{32} - \beta_{31}\beta_{12} = \Delta_3(\mu_1^{-1} + \mu_2^{-1}) \quad (\text{B.8})$$

$$\Delta_1 = \beta_{21}\beta_{32} - \beta_{31}\beta_{22} = \Delta_3\mu_1^{-1}\mu_2^{-1} \quad (\text{B.9})$$

Further, using the above relations, the discriminant of (B.5) can be expressed as

$$D \triangleq \Delta_2^2 - 4\Delta_1\Delta_3 = \Delta_3^2(\mu_1^{-1} - \mu_2^{-1})^2. \quad (\text{B.10})$$

Thus, the roots of (B.5) are given by

$$\gamma_i = \frac{\Delta_3(\mu_1^{-1} + \mu_2^{-1}) \pm \Delta_3(\mu_1^{-1} - \mu_2^{-1})}{2\Delta_3\mu_1^{-1}\mu_2^{-1}} = \mu_1 \text{ or } \mu_2 \quad (\text{B.11})$$

Here,  $\gamma_1 = \mu_2$  corresponds to + sign and  $\gamma_2 = \mu_1$  to - sign.

Now, with the help of the equalities (B.7)–(B.11), we consider the estimated case. From (A.1), we have the estimated eigenvectors corresponding to the signal subspace as

$$\hat{\mathbf{e}}_i = k_i \left[ \beta_i + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^M w_{ji} \beta_j \right] + o(1/N^2), \quad i = 1, 2, \dots, K. \quad (\text{B.12})$$

where  $k_i$  is defined as (A.2). In a two-source scene, the generalized eigenvalues of the estimated matrix pencil  $\{\mathbf{E}_1, \mathbf{E}_2\}$  given by

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = \hat{\gamma} \begin{bmatrix} e_{21} & e_{22} \\ e_{31} & e_{32} \end{bmatrix}. \quad (\text{B.13})$$

Since  $\mathbf{e}_i = \hat{\mathbf{e}}_i e^{j\phi_i}$ ,  $i = 1, 2, \dots, K$ , the above equation reduces to

$$\begin{vmatrix} \hat{c}_{11} - \hat{\gamma}\hat{c}_{21} & \hat{c}_{12} - \hat{\gamma}\hat{c}_{22} \\ \hat{c}_{21} - \hat{\gamma}\hat{c}_{31} & \hat{c}_{22} - \hat{\gamma}\hat{c}_{32} \end{vmatrix} = \hat{\Delta}_1\hat{\gamma}^2 - \hat{\Delta}_2\hat{\gamma} + \hat{\Delta}_3 = 0 \quad (\text{B.14})$$

In (B.13) and (B.14),  $e_{ij}$  and  $\hat{c}_{ij}$  represents the  $i^{\text{th}}$  elements of the estimated eigen-

vectors  $\mathbf{e}_j$  and  $\hat{\mathbf{e}}_j$ , respectively. Following (B.14), define  $\hat{\Delta}_1, \hat{\Delta}_2, \hat{\Delta}_3$  as

$$\hat{\Delta}_1 \triangleq \begin{vmatrix} \hat{e}_{21} & \hat{e}_{22} \\ \hat{e}_{31} & \hat{e}_{32} \end{vmatrix}, \quad \hat{\Delta}_2 \triangleq \begin{vmatrix} \hat{e}_{11} & \hat{e}_{12} \\ \hat{e}_{31} & \hat{e}_{32} \end{vmatrix}, \quad \hat{\Delta}_3 \triangleq \begin{vmatrix} \hat{e}_{11} & \hat{e}_{12} \\ \hat{e}_{21} & \hat{e}_{22} \end{vmatrix}. \quad (\text{B.15})$$

To find a first-order approximation for  $\hat{\Delta}_1, \hat{\Delta}_2$  and  $\hat{\Delta}_3$ , we begin with the interelement multiplication of  $\hat{\mathbf{e}}_i, i = 1, 2$ . This can be written as

$$\begin{aligned} \hat{e}_{p1}\hat{e}_{q2} = k_1 k_2 & \left[ \beta_{p1}\beta_{q2} + \frac{1}{\sqrt{N}} \left[ \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1}\beta_{pj}\beta_{q2} + \sum_{\substack{j=1 \\ j \neq 2}}^M w_{j2}\beta_{qj}\beta_{p1} \right] \right. \\ & \left. + \frac{1}{N} \sum_{\substack{k=1 \\ k \neq 1}}^M \sum_{\substack{l=1 \\ l \neq 2}}^M w_{k1}w_{l2}\beta_{pk}\beta_{ql} \right] + o(1/N^2). \end{aligned} \quad (\text{B.16})$$

With (B.16) in (B.15), we have

$$\begin{aligned} \hat{\Delta}_1 &= \hat{e}_{21}\hat{e}_{32} - \hat{e}_{31}\hat{e}_{22} \\ &= k_1 k_2 \left[ \Delta_1 + \frac{1}{\sqrt{N}} \left[ \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1}(\beta_{2j}\beta_{32} - \beta_{3j}\beta_{22}) + \sum_{\substack{j=1 \\ j \neq 2}}^M w_{j2}(\beta_{3j}\beta_{21} - \beta_{2j}\beta_{31}) \right] \right. \\ & \quad \left. + \frac{1}{N} \sum_{\substack{k=1 \\ k \neq 1}}^M \sum_{\substack{l=1 \\ l \neq 2}}^M w_{k1}w_{l2}(\beta_{2k}\beta_{3l} - \beta_{3k}\beta_{2l}) \right] + o(1/N^2). \end{aligned} \quad (\text{B.17})$$

Defining

$$\Delta_{ijkl} \triangleq \beta_{ik}\beta_{jl} - \beta_{jk}\beta_{il} \quad (\text{B.18})$$

(B.17) can be written as

$$\hat{\Delta}_1 = k_1 k_2 \left[ \Delta_1 + \frac{1}{\sqrt{N}} \Gamma_{11} + \frac{1}{N} \Gamma_{21} \right] + o(1/N^2) \quad (\text{B.19})$$

where

$$\Gamma_{11} = \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \Delta_{23j2} + \sum_{\substack{j=1 \\ j \neq 2}}^M w_{j2} \Delta_{32j1} \quad (\text{B.20})$$

$$\Gamma_{21} = \sum_{\substack{k=1 \\ k \neq 1}}^M \sum_{\substack{l=1 \\ l \neq 2}}^M w_{k1} w_{l2} \Delta_{23kl} \quad (\text{B.21})$$

Similarly, we have

$$\hat{\Delta}_2 = k_1 k_2 \left[ \Delta_2 + \frac{1}{\sqrt{N}} \Gamma_{12} + \frac{1}{N} \Gamma_{22} \right] + o(1/N^2) \quad (\text{B.22})$$

$$\hat{\Delta}_3 = k_1 k_2 \left[ \Delta_3 + \frac{1}{\sqrt{N}} \Gamma_{13} + \frac{1}{N} \Gamma_{23} \right] + o(1/N^2) \quad (\text{B.23})$$

where

$$\Gamma_{12} = \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \Delta_{13j2} + \sum_{\substack{j=1 \\ j \neq 2}}^M w_{j2} \Delta_{31j1} \quad (\text{B.24})$$

$$\Gamma_{13} = \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \Delta_{12j2} + \sum_{\substack{j=1 \\ j \neq 2}}^M w_{j2} \Delta_{21j1} \quad (\text{B.25})$$

$$\Gamma_{22} = \sum_{\substack{k=1 \\ k \neq 1}}^M \sum_{\substack{l=1 \\ l \neq 2}}^M w_{k1} w_{l2} \Delta_{13kl} ; \quad \Gamma_{23} = \sum_{\substack{k=1 \\ k \neq 1}}^M \sum_{\substack{l=1 \\ l \neq 2}}^M w_{k1} w_{l2} \Delta_{12kl} \quad (\text{B.26})$$

The generalized eigenvalue of (B.14) within a first-order approximation can be found by using (B.19) – (B.26). To see this, let

$$\Gamma_i \triangleq \frac{1}{\sqrt{N}} \Gamma_{1i} + \frac{1}{N} \Gamma_{2i} \quad , \quad i = 1, 2, 3, \quad (\text{B.27})$$

and with (B.14), (B.19), (B.22) and (B.23), the generalized eigenvalues have the form



$$\hat{\gamma}_i = \frac{\hat{\Delta}_2 \pm (\hat{\Delta}_2^2 - 4\hat{\Delta}_1\hat{\Delta}_3)^{1/2}}{2\hat{\Delta}_1}$$

$$= \frac{(\Delta_2 + \Gamma_2) \pm \left[ \Delta_2^2 - 4\Delta_1\Delta_3 + 2\Delta_2\Gamma_2 + \Gamma_2^2 - 4(\Delta_3\Gamma_1 + \Delta_1\Gamma_3 + \Gamma_1\Gamma_3) \right]^{1/2}}{2\Delta_1(1 + \Gamma_1/\Delta_1)} \quad (\text{B.28})$$

Here,  $\Gamma_i$  defined in (B.27) represents the perturbation of  $\hat{\Delta}_i$  from  $\Delta_i$  and for large  $N$ , from (B.19) and (B.27) noticing that  $|\Gamma_1/\Delta_1| \ll 1$ , we can rewrite (B.28) as

$$\hat{\gamma}_i \approx \frac{1}{2\Delta_1} \left[ (\Delta_2 + \Gamma_2) \pm \left( D \left[ 1 + \frac{2\Delta_2\Gamma_2 + \Gamma_2^2 - 4(\Delta_3\Gamma_1 + \Delta_1\Gamma_3 + \Gamma_1\Gamma_3)}{D} \right] \right)^{1/2} \right]$$

$$\times \left[ 1 - \frac{\Gamma_1}{\Delta_1} + \frac{\Gamma_1^2}{\Delta_1^2} \right] \quad (\text{B.29})$$

where  $D$  is given by (B.10). Once again, for large  $N$  we have

$$\left| \frac{2\Delta_2\Gamma_2 + \Gamma_2^2 - 4(\Delta_3\Gamma_1 + \Delta_1\Gamma_3 + \Gamma_1\Gamma_3)}{D} \right| < 1 \quad (\text{B.30})$$

which allows one to further approximate (B.29) as

$$\hat{\gamma}_i \approx \frac{\Delta_2 \pm \sqrt{D}}{2\Delta_1} \left[ 1 - \frac{\Gamma_1}{\Delta_1} + \frac{\Gamma_1^2}{\Delta_1^2} \right]$$

$$+ \frac{2\sqrt{D}\Gamma_2 \pm [2\Delta_2\Gamma_2 + \Gamma_2^2 - 4(\Delta_3\Gamma_1 + \Delta_1\Gamma_3 + \Gamma_1\Gamma_3)]}{4\Delta_1\sqrt{D}} \left[ 1 - \frac{\Gamma_1}{\Delta_1} + \frac{\Gamma_1^2}{\Delta_1^2} \right]$$

$$\mp \frac{[2\Delta_2\Gamma_2 + \Gamma_2^2 - 4(\Delta_3\Gamma_1 + \Delta_1\Gamma_3 + \Gamma_1\Gamma_3)]^2}{16\Delta_1 D \sqrt{D}} \left[ 1 - \frac{\Gamma_1}{\Delta_1} + \frac{\Gamma_1^2}{\Delta_1^2} \right] \quad (\text{B.31})$$

Neglecting terms formed by combinations of triple products of  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_3$ , (B.31) reduces to

$$\begin{aligned} \hat{\gamma}_i \approx \gamma_i \left(1 - \frac{\Gamma_1}{\Delta_1} + \frac{\Gamma_1^2}{\Delta_1^2}\right) + \frac{(\sqrt{D} \pm \Delta_2)\Gamma_2 \mp 2(\Delta_3\Gamma_1 + \Delta_1\Gamma_3)}{2\Delta_1\sqrt{D}} \\ - \frac{(\sqrt{D} \pm \Delta_2)\Gamma_1\Gamma_2 \mp 2(\Delta_3\Gamma_1^2 + \Delta_1\Gamma_2^2/4)}{2\Delta_1^2\sqrt{D}} \mp \frac{[\Delta_2\Gamma_2 - 2\Delta_3\Gamma_1 - 2\Delta_1\Gamma_3]^2}{4\Delta_1 D\sqrt{D}} \end{aligned} \quad (B.32)$$

Finally, using (B.27) and retaining only those terms of order greater than or equal to  $1/N$ , we have

$$\hat{\gamma}_i \approx \gamma_i + \frac{1}{\sqrt{N}} \Phi_{1i} + \frac{1}{N} \Phi_{2i} + o\left(\frac{1}{N\sqrt{N}}\right) \quad (B.33)$$

where

$$\Phi_{1i} = -\frac{\gamma_i}{\Delta_1} \Gamma_{11} + \frac{(\sqrt{D} \pm \Delta_2)\Gamma_{12} \mp 2(\Delta_3\Gamma_{11} + \Delta_1\Gamma_{13})}{2\Delta_1\sqrt{D}} \quad (B.34)$$

and

$$\begin{aligned} \Phi_{2i} = -\frac{\gamma_i}{\Delta_1} \Gamma_{21} + \frac{\gamma_i}{\Delta_1^2} \Gamma_{11}^2 + \frac{(\sqrt{D} \pm \Delta_2)\Gamma_{22} \mp 2(\Delta_3\Gamma_{21} + \Delta_1\Gamma_{23})}{2\Delta_1\sqrt{D}} \\ - \frac{(\sqrt{D} \pm \Delta_2)\Gamma_{11}\Gamma_{12} \mp 2(\Delta_3\Gamma_{11}^2 + \Delta_1\Gamma_{12}^2/4)}{2\Delta_1^2\sqrt{D}} \mp \frac{[\Delta_2\Gamma_{12} - 2\Delta_3\Gamma_{11} - 2\Delta_1\Gamma_{13}]^2}{4\Delta_1 D\sqrt{D}} \end{aligned} \quad (B.35)$$

$\Phi_{1i}$  and  $\Phi_{2i}$  in (B.34)–(B.35) can be further simplified with the help of (B.7)–(B.11).

For  $i = 1$  and  $\gamma_1 = \mu_2$ , we have

$$\Phi_{11} = -\frac{1}{\Delta_3(\mu_1^{-1} - \mu_2^{-1})} [\mu_2^2 \Gamma_{11} - \mu_2 \Gamma_{12} + \Gamma_{13}] \quad (B.36)$$

and

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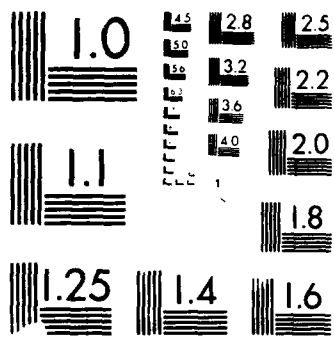
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$$\begin{aligned} \Phi_{21} = & -\frac{\mu_2^2 \Gamma_{21} - \mu_2 \Gamma_{22} + \Gamma_{23}}{\Delta_3(\mu_1^{-1} - \mu_2^{-1})} + \frac{\mu_2^2 \Gamma_{11}^2 - \mu_2 \Gamma_{11} \Gamma_{12} + \Gamma_{12}^2/4}{\Delta_3^2 \mu_1^{-1} \mu_2^{-1} (\mu_1^{-1} - \mu_2^{-1})} \\ & - \frac{[(\mu_1^{-1} + \mu_2^{-1}) \Gamma_{12} - 2\Gamma_{11} - 2\mu_1^{-1} \mu_2^{-1} \Gamma_{13}]^2}{4\Delta_3^2 \mu_1^{-1} \mu_2^{-1} (\mu_1^{-1} - \mu_2^{-1})^3} \end{aligned} \quad (B.37)$$

Similarly, for  $i = 2$  and  $\gamma_2 = \mu_2$  we have

$$\Phi_{12} = \frac{1}{\Delta_3(\mu_1^{-1} - \mu_2^{-1})} [\mu_1^2 \Gamma_{11} - \mu_1 \Gamma_{12} + \Gamma_{13}] \quad (B.38)$$

$$\begin{aligned} \Phi_{22} = & \frac{\mu_1^2 \Gamma_{21} - \mu_1 \Gamma_{22} + \Gamma_{23}}{\Delta_3(\mu_1^{-1} - \mu_2^{-1})} - \frac{\mu_1^2 \Gamma_{11}^2 - \mu_1 \Gamma_{11} \Gamma_{12} + \Gamma_{12}^2/4}{\Delta_3^2 \mu_1^{-1} \mu_2^{-1} (\mu_1^{-1} - \mu_2^{-1})} \\ & + \frac{[(\mu_1^{-1} + \mu_2^{-1}) \Gamma_{12} - 2\Gamma_{11} - 2\mu_1^{-1} \mu_2^{-1} \Gamma_{13}]^2}{4\Delta_3^2 \mu_1^{-1} \mu_2^{-1} (\mu_1^{-1} - \mu_2^{-1})^3} \end{aligned} \quad (B.39)$$

From now on, with the statistical properties derived in [16, 17], we compute the mean and variance of the estimated generalized eigenvalues. Consider the first generalized eigenvalue,  $\hat{\gamma}_1$ . Then, from (B.33), the expected value of  $\hat{\gamma}_1$  is

$$E[\hat{\gamma}_1] = \gamma_1 + \frac{1}{\sqrt{N}} E[\Phi_{11}] + \frac{1}{N} E[\Phi_{21}] + o\left(\frac{1}{N\sqrt{N}}\right). \quad (B.40)$$

With (B.20), (B.24) and (B.25), it is easy to verify that  $E[\Phi_{11}]$  consists of the first-order moments of  $w_{ij}$  which gives  $E[\Phi_{11}] = 0$  and hence

$$\begin{aligned} E[\hat{\gamma}_1] = & \gamma_1 + \frac{1}{N} E[\Phi_{21}] + o\left(\frac{1}{N\sqrt{N}}\right) \\ = & \gamma_1 - \frac{E[\mu_2^2 \Gamma_{21} - \mu_2 \Gamma_{22} + \Gamma_{23}]}{N \Delta_3(\mu_1^{-1} - \mu_2^{-1})} + \frac{E[\mu_2^2 \Gamma_{11}^2 - \mu_2 \Gamma_{11} \Gamma_{12} + \Gamma_{12}^2/4]}{N \Delta_3^2 \mu_1^{-1} \mu_2^{-1} (\mu_1^{-1} - \mu_2^{-1})} \\ & - \frac{E[(\mu_1^{-1} + \mu_2^{-1}) \Gamma_{12} - 2\Gamma_{11} - 2\mu_1^{-1} \mu_2^{-1} \Gamma_{13}]^2}{4N \Delta_3^2 \mu_1^{-1} \mu_2^{-1} (\mu_1^{-1} - \mu_2^{-1})^3} + o\left(\frac{1}{N\sqrt{N}}\right). \end{aligned} \quad (B.41)$$

With (B.21), (B.26) and (B.26), the terms inside the first expected operator can be written as

$$\mu_2^2 \Gamma_{21} - \mu_2 \Gamma_{22} + \Gamma_{23} = \sum_{\substack{k=1 \\ k \neq 1}}^M \sum_{\substack{l=1 \\ l \neq 2}}^M w_{k1} w_{l2} (\mu_2^2 \Delta_{23kl} - \mu_2 \Delta_{13kl} + \Delta_{12kl}). \quad (\text{B.42})$$

With (A.4) in (B.42) we also have

$$\begin{aligned} E[\mu_2^2 \Gamma_{21} - \mu_2 \Gamma_{22} + \Gamma_{23}] &= \sum_{\substack{k=1 \\ k \neq 1}}^M \sum_{\substack{l=1 \\ l \neq 2}}^M E[w_{k1} w_{l2}] (\mu_2^2 \Delta_{23kl} - \mu_2 \Delta_{13kl} + \Delta_{12kl}) \\ &= \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)^2} (\mu_2^2 \Delta_{2321} - \mu_2 \Delta_{1321} + \Delta_{1221}) = 0 \end{aligned} \quad (\text{B.43})$$

since, from (B.18), (B.7)–(B.9), it is easy to show that  $\Delta_{2321} = -\Delta_1$ ,  $\Delta_{1321} = -\Delta_2$  and  $\Delta_{1221} = -\Delta_3$ .

Also, with (B.20) and (B.24), we can write

$$\begin{aligned} \mu_2 \Gamma_{11} - \Gamma_{12}/2 &= \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} (\mu_2 \Delta_{23j2} - \Delta_{13j2}/2) + \sum_{\substack{j=1 \\ j \neq 2}}^M w_{j2} (\mu_2 \Delta_{32j1} - \Delta_{31j1}/2) \\ &= \sum_{j=3}^M \left\{ w_{j1} (\mu_2 \Delta_{23j2} - \Delta_{13j2}/2) + w_{j2} (\mu_2 \Delta_{32j1} - \Delta_{31j1}/2) \right\} \end{aligned} \quad (\text{B.44})$$

since from (B.18)

$$\Delta_{ijkl} = 0 \quad \text{for } k = l. \quad (\text{B.45})$$

Hence, together with (B.44) and (A.4), we can easily show that

$$E[\mu_2^2 \Gamma_{11}^2 - \mu_2 \Gamma_{11} \Gamma_{12} + \Gamma_{12}^2/4] = 0. \quad (\text{B.46})$$

Similarly, for the expected value of the last term in (B.39), with (B.20), (B.24) and (B.25) we can write

$$\begin{aligned}
 & (\mu_1^{-1} + \mu_2^{-1})\Gamma_{12} - 2\Gamma_{11} - 2\mu_1^{-1}\mu_2^{-1}\Gamma_{13} \\
 &= \sum_{\substack{j=1 \\ j \neq 1}}^M w_{j1} \left[ (\mu_1^{-1} + \mu_2^{-1})\Delta_{13j2} - 2\Delta_{23j2} - 2\mu_1^{-1}\mu_2^{-1}\Delta_{12j2} \right] \\
 &+ \sum_{\substack{j=1 \\ j \neq 2}}^M w_{j2} \left[ (\mu_1^{-1} + \mu_2^{-1})\Delta_{31j1} - 2\Delta_{32j1} - 2\mu_1^{-1}\mu_2^{-1}\Delta_{21j1} \right] \\
 &= \sum_{j=3}^M \left[ w_{j1} \left[ (\mu_1^{-1} + \mu_2^{-1})\Delta_{13j2} - 2\Delta_{23j2} - 2\mu_1^{-1}\mu_2^{-1}\Delta_{12j2} \right] \right. \\
 &\quad \left. + w_{j2} \left[ (\mu_1^{-1} + \mu_2^{-1})\Delta_{31j1} - 2\Delta_{32j1} - 2\mu_1^{-1}\mu_2^{-1}\Delta_{21j1} \right] \right] \quad (B.47)
 \end{aligned}$$

Here, once again we use the fact of (B.45). Then, with (B.47) and using (A.4), it can also be shown that

$$E \left[ [(\mu_1^{-1} + \mu_2^{-1})\Gamma_{12} - 2\Gamma_{11} - 2\mu_1^{-1}\mu_2^{-1}\Gamma_{13}]^2 \right] = 0. \quad (B.48)$$

Finally, with (B.43), (B.46) and (B.48) in (B.39), we obtain

$$E[\hat{\gamma}_1] = \gamma_1 + o\left(\frac{1}{N\sqrt{N}}\right) \quad (B.49)$$

Thus, within a first-order approximation, the sample generalized eigenvalue  $\hat{\gamma}_1$  is an unbiased estimator for  $\gamma_1$ .

With the help of (B.33), (B.36), (B.20), (B.24)–(B.25), and some algebra, it is easy to compute their variances. Now, with (A.3)

$$Var(\hat{\gamma}_1) = \frac{1}{N} E[|\Phi_{11}|^2]$$

$$\begin{aligned}
 &= \frac{1}{N |\Delta_3|^2 |\mu_1^{-1} - \mu_2^{-1}|^2} \left[ \sum_{\substack{i=1 \\ i \neq 1}}^M \sum_{\substack{j=1 \\ j \neq 1}}^M E[w_{i1} w_{j1}^*] I_i I_j^* \right. \\
 &+ \sum_{\substack{i=1 \\ i \neq 1}}^M \sum_{\substack{j=1 \\ j \neq 2}}^M E[w_{i1} w_{j2}^*] I_i II_j^* + \sum_{\substack{i=1 \\ i \neq 2}}^M \sum_{\substack{j=1 \\ j \neq 1}}^M E[w_{i2} w_{j1}^*] II_i I_j^* + \sum_{\substack{i=1 \\ i \neq 2}}^M \sum_{\substack{j=1 \\ j \neq 2}}^M E[w_{i2} w_{j2}^*] II_i II_j^* \left. \right] \\
 &= \frac{1}{N |\Delta_3|^2 |\mu_1^{-1} - \mu_2^{-1}|^2} \left[ \sum_{\substack{i=1 \\ i \neq 1}}^M \frac{\lambda_1 \lambda_i}{(\lambda_1 - \lambda_i)^2} |I_i|^2 + \sum_{\substack{i=1 \\ i \neq 2}}^M \frac{\lambda_2 \lambda_i}{(\lambda_2 - \lambda_i)^2} |II_i|^2 \right] \quad (B.50)
 \end{aligned}$$

where

$$I_j = \mu_2^2 \Delta_{23j2} - \mu_2 \Delta_{13j2} + \Delta_{12j2} \quad (B.51)$$

$$II_j = \mu_2^2 \Delta_{32j1} - \mu_2 \Delta_{31j1} + \Delta_{21j1}. \quad (B.52)$$

Also, from the definition of  $\Delta_{ijkl}$  in (B.18) we notice that  $I_j$  in (B.51) is zero for  $j = 2$  and  $II_j$  in (B.52) zero for  $j = 1$ . Therefore, with  $\lambda_3 = \lambda_4 = \dots = \lambda_M = \sigma^2$ , (B.50) is simplified to

$$Var(\hat{\gamma}_1) = \frac{1}{N |\Delta_3|^2 |\mu_1^{-1} - \mu_2^{-1}|^2} \sum_{i=3}^M \left[ \frac{\lambda_1 \sigma^2}{(\lambda_1 - \sigma^2)^2} |I_i|^2 + \frac{\lambda_2 \sigma^2}{(\lambda_2 - \sigma^2)^2} |II_i|^2 \right] \quad (B.53)$$

Now, we evaluate  $I_i$  and  $II_i$  from (B.18) and (B.1). First,

$$\begin{aligned}
 I_i &= \mu_2^2 \Delta_{23i2} - \mu_2 \Delta_{13i2} + \Delta_{12i2} \\
 &= \mu_2^2 (\beta_{2i} \beta_{32} - \beta_{3i} \beta_{22}) - \mu_2 (\beta_{1i} \beta_{32} - \beta_{3i} \beta_{12}) + (\beta_{1i} \beta_{22} - \beta_{2i} \beta_{12}) \\
 &= \frac{\mu_1^{(M-1)/2} \mu_2^2 (\mu_2^{-1} - \mu_1^{-1})}{\sqrt{2M(1 - |\rho_s|)}} [\mu_1^{-1} \mu_2^{-1} \beta_{1i} - (\mu_2^{-1} + \mu_1^{-1}) \beta_{2i} + \beta_{3i}] \quad (B.54)
 \end{aligned}$$



Also, from (B.52)

$$\begin{aligned} H_i &= \mu_2^2 \Delta_{32i1} - \mu_2 \Delta_{31i1} + \Delta_{21i1} \\ &= \frac{-\mu_1^{(M-1)/2} \mu_2^2 (\mu_2^{-1} - \mu_1^{-1})}{\sqrt{2M(1+|\rho_s|)}} [\mu_1^{-1} \mu_2^{-1} \beta_{1i} - (\mu_2^{-1} + \mu_1^{-1}) \beta_{2i} + \beta_{3i}]. \end{aligned} \quad (B.55)$$

(B.53) together with (B.54), (B.55) gives

$$\begin{aligned} Var(\hat{\gamma}_1) &= \frac{1}{N|\Delta_3|^2} \left[ \frac{\lambda_1 \sigma^2}{(\lambda_1 - \sigma^2)^2} \frac{1}{2M(1-|\rho_s|)} + \frac{\lambda_2 \sigma^2}{(\lambda_2 - \sigma^2)^2} \frac{1}{2M(1+|\rho_s|)} \right] \\ &\quad \sum_{i=3}^M |\mu_1^{-1} \mu_2^{-1} \beta_{1i} - (\mu_2^{-1} + \mu_1^{-1}) \beta_{2i} + \beta_{3i}|^2 \end{aligned} \quad (B.56)$$

Replacing  $\Delta_3$  with (B.7), this gives

$$\begin{aligned} Var(\hat{\gamma}_1) &= \frac{M}{2N|\mu_1^{-1} - \mu_2^{-1}|^2} \left[ (1+|\rho_s|) \frac{\lambda_1 \sigma^2}{(\lambda_1 - \sigma^2)^2} + (1-|\rho_s|) \frac{\lambda_2 \sigma^2}{(\lambda_2 - \sigma^2)^2} \right] \\ &\quad \sum_{i=3}^M |\mu_1^{-1} \mu_2^{-1} \beta_{1i} - (\mu_2^{-1} + \mu_1^{-1}) \beta_{2i} + \beta_{3i}|^2. \end{aligned} \quad (B.57)$$

To simplify this further, notice that

$$\sum_{k=1}^M \beta_{ik} \beta_{jk}^* = \delta_{ij}, \quad i = 1, 2, \dots, M \text{ and } j = 1, 2, \dots, M. \quad (B.58)$$

For  $i = j = 1$ , (B.58) together with (B.1) gives

$$\sum_{k=3}^M |\beta_{1k}|^2 = 1 - \frac{2}{M(1-|\rho_s|^2)} \left[ 1 - |\rho_s| \operatorname{Re}(\mu_1^{(M-1)/2} \mu_2^{-(M-1)/2}) \right]. \quad (B.59)$$

Similarly,

$$\sum_{k=3}^M |\beta_{2k}|^2 = 1 - \frac{2}{M(1 - |\rho_s|^2)} \left[ 1 - |\rho_s| \operatorname{Re}(\mu_1^{(M-3)/2} \mu_2^{-(M-3)/2}) \right] \quad (\text{B.60})$$

and

$$\sum_{k=3}^M |\beta_{3k}|^2 = 1 - \frac{2}{M(1 - |\rho_s|^2)} \left[ 1 - |\rho_s| \operatorname{Re}(\mu_1^{(M-5)/2} \mu_2^{-(M-5)/2}) \right]. \quad (\text{B.61})$$

Also, for  $i = 1, j = 2$  we have

$$\sum_{k=3}^M \beta_{1k} \beta_{2k}^* = - \frac{(\mu_1 + \mu_2) - |\rho_s| (\mu_1^{-(M-3)/2} \mu_2^{(M-1)/2} + \mu_1^{(M-1)/2} \mu_2^{-(M-3)/2})}{M(1 - |\rho_s|^2)} \quad (\text{B.62})$$

$$\sum_{k=3}^M \beta_{2k} \beta_{3k}^* = - \frac{(\mu_1 + \mu_2) - |\rho_s| (\mu_1^{-(M-5)/2} \mu_2^{(M-3)/2} + \mu_1^{(M-3)/2} \mu_2^{-(M-5)/2})}{M(1 - |\rho_s|^2)} \quad (\text{B.63})$$

and

$$\sum_{k=3}^M \beta_{3k} \beta_{1k}^* = - \frac{(\mu_1^{-2} + \mu_2^{-2}) - |\rho_s| (\mu_1^{-(M-1)/2} \mu_2^{(M-5)/2} + \mu_1^{(M-5)/2} \mu_2^{-(M-1)/2})}{M(1 - |\rho_s|^2)} \quad (\text{B.64})$$

With (B.59) – (B.64), the last term in (B.57) can be written as

$$\begin{aligned} & \sum_{i=3}^M |\mu_1^{-1} \mu_2^{-1} \beta_{1i} - (\mu_2^{-1} + \mu_1^{-1}) \beta_{2i} + \beta_{3i}|^2 \\ &= \sum_{i=3}^M \left[ |\beta_{1i}|^2 + |\mu_1^{-1} + \mu_2^{-1}|^2 |\beta_{2i}|^2 + |\beta_{3i}|^2 - 2 \operatorname{Re}[\mu_1^{-1} \mu_2^{-1} (\mu_1 + \mu_2) \beta_{1i} \beta_{2i}^* \right. \\ & \quad \left. + (\mu_1^{-1} + \mu_2^{-1}) \beta_{2i} \beta_{3i}^* - \mu_1 \mu_2 \beta_{3i} \beta_{1i}^*] \right] \\ &= 4 + 2 \operatorname{Re}(\mu_1^{-1} \mu_2). \end{aligned} \quad (\text{B.65})$$

Finally, using (B.65), (B.57) reduces to

$$Var(\hat{\gamma}_1) = \frac{M \sigma^2 (2 + \text{Re}(\mu_1^{-1} \mu_2))}{2N(1 - \text{Re}(\mu_1^{-1} \mu_2))} \left[ \frac{\lambda_1}{(\lambda_1 - \sigma^2)^2} (1 + |\rho_s|) + \frac{\lambda_2}{(\lambda_2 - \sigma^2)^2} (1 - |\rho_s|) \right]. \quad (\text{B.66})$$

The mean and variance of  $\hat{\gamma}_2$  are identically the same as that of  $\hat{\gamma}_1$  and the covariance between  $\hat{\gamma}_1$  and  $\hat{\gamma}_2$  is given by

$$Cov(\hat{\gamma}_1, \hat{\gamma}_2) = - \frac{M \sigma^2 (2 + \text{Re}(\mu_1^{-1} \mu_2))}{2N(1 - \text{Re}(\mu_1^{-1} \mu_2))} \mu_1^{(M-5)/2} \mu_2^{-(M-5)/2} \left[ \frac{\lambda_1}{(\lambda_1 - \sigma^2)^2} (1 + |\rho_s|) - \frac{\lambda_2}{(\lambda_2 - \sigma^2)^2} (1 - |\rho_s|) \right]. \quad (\text{B.67})$$

### Appendix C

#### Bias Constants in a Two-Source Scene

$$\begin{aligned} p_1 = & (3M^3 - 12M^2 + 15M - 6) + \\ & (0.704M^5 - 4.206M^4 + 8.793M^3 - 4.195M^2 - 27.194M + 81.592)\omega_d^2 + \\ & (0.0642M^7 - 0.42M^6 + 0.945M^5 + 0.377M^4 - 6.937M^3 + 53.93M^2 \\ & - 61.517M + 12.417)\omega_d^4 + o(\omega_d^6) \\ q_1 = & (3M^4 - 18M^3 + 39M^2 - 36M + 12) + \\ & (0.2M^6 - 2.6M^5 + 11.45M^4 - 29.3M^3 + 38.95M^2 + 46.5M - 212.98)\omega_d^2 + \\ & (0.078M^8 - 0.804M^7 + 3.54M^6 - 9.48M^5 + 18.66M^4 - 33.37M^3 + 61.32M^2 \\ & - 133.61M + 199.34)\omega_d^4 + o(\omega_d^6) \\ r_1 = & (-3M^4 + 15M^3 - 27M^2 + 21M - 6) + \\ & (-0.2M^6 + 1.9M^5 - 6.88M^4 + 14.09M^3 - 10M^2 - 69.1M + 200.63)\omega_d^2 + \\ & (-0.053M^8 + 0.492M^7 - 1.884M^6 + 1.535M^5 + 10.81M^4 - 45.02M^3 + 78.99M^2 \\ & - 26.29M - 80.31)\omega_d^4 + o(\omega_d^6) \\ p_2 = & (6M^3 - 24M^2 + 30M - 12) + \\ & (-0.602M^5 + 3.606M^4 - 14.406M^3 + 33.601M^2 - 107.398M + 229.2)\omega_d^2 + \\ & (0.2572M^7 - 1.89M^6 + 6.235M^5 - 10.309M^4 - 0.697M^3 + 50.68M^2 \\ & - 117.78M + 70.12)\omega_d^4 + o(\omega_d^6) \end{aligned}$$

$$\begin{aligned} q_2 = & (6M^4 - 36M^3 + 78M^2 - 72M + 24) + \\ & (-0.602M^6 + 3.806M^5 - 9.115M^4 + 11.41M^3 - 27.6M^2 + 186M - 445.96)\omega_d^2 + \\ & (0.173M^8 - 1.702M^7 + 3.988M^6 - 17.221M^5 + 30.96M^4 - 64.43M^3 + 144.98M^2 \\ & - 264.61M + 398.12)\omega_d^4 + o(\omega_d^6) \end{aligned}$$

$$\begin{aligned} r_2 = & (-6M^4 + 30M^3 - 54M^2 + 42M - 12) + \\ & (0.602M^6 - 3.213M^5 + 5.51M^4 - 3.833M^3 + 42.233M^2 - 116.25M + 493.5)\omega_d^2 + \\ & (-0.173M^8 + 1.446M^7 - 5.035M^6 + 5.226M^5 + 17.42M^4 - 81.96M^3 + 167.84M^2 \\ & - 134.45M - 102.28)\omega_d^4 + o(\omega_d^6) \end{aligned}$$

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